

# THREE LECTURES ON COX RINGS

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## INTRODUCTION

These are notes of an introductory course held at the conference “Torsors: Theory and Applications” in Edinburgh, January 2011. Cox rings play meanwhile an important role in Arithmetic and Algebraic Geometry, and in fact appeared independently in these fields [12, 13, 25]. We present basic ideas and concepts, in particular, we treat the interaction with Geometric Invariant Theory. The notes are kept as a survey; for details and proofs we refer to [4].

The first lecture begins with a rigorous definition of the Cox sheaf  $\mathcal{R}$  of a variety  $X$  as a sheaf of algebras graded by the divisor class group  $\mathrm{Cl}(X)$ . The Cox ring is then the algebra  $\mathcal{R}(X)$  of global sections. After recalling the basic correspondence between graded algebras and quasitorus actions, we discuss the relative spectrum  $\hat{X} := \mathrm{Spec}_X \mathcal{R}$  of the Cox sheaf. We call  $\hat{X} \rightarrow X$  the characteristic space; it coincides with the universal torsor [12, 38] precisely for locally factorial varieties  $X$ . The first lecture ends with a characterization of  $\hat{X}$  in terms of Geometric Invariant Theory.

The aim of the second lecture is to present a machinery for encoding varieties via their Cox ring. A first input is an explicit description of the variation of not necessarily quasiprojective good quotients for quasitorus actions on affine varieties. This allows to encode torically embeddable varieties with finitely generated Cox ring in terms of what we call “bunched rings”. Specialized to the case of toric varieties [15, 33, 18], the bunched ring data correspond to fans via Gale duality. Moreover, the language of bunched rings extends basic features of techniques developed for weighted complete intersections [17, 26] to the multigraded case. We show how to read off basic geometric properties from the defining data and briefly touch the relations to Mori Theory.

The third Lecture is devoted to varieties with torus action. Generalizing the case of a toric variety, we describe the Cox ring in terms of the action. In the case of a complexity one action, one obtains a very explicit presentation in terms of trinomial relations. Combining this with the language of bunched rings leads to a concrete description of rational complete varieties with a complexity one action turning them into an easy accessible class for concrete computations. We demonstrate this in the case of  $\mathbb{K}^*$ -surfaces. Some classification results on Fano threefolds and del Pezzo surfaces based on this approach are included in the text.

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## 1. FIRST LECTURE

**1.1. Cox sheaves and Cox rings.** We work in the category of (reduced) varieties over an algebraically closed field  $\mathbb{K}$  of characteristic zero. The Cox ring  $\mathcal{R}(X)$  will contain a lot of accessible information about the underlying variety  $X$ , the rough idea is to define it as

$$\mathcal{R}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

where the grading is by the divisor class group  $\text{Cl}(X)$  of  $X$ . A priori, it is not clear what the ring structure should be, in particular if there is torsion in the divisor class group  $\text{Cl}(X)$ . Our first task is to clarify this; we closely follow [4, Chap. I].

We begin with recalling the basic notions on divisors on the normal algebraic variety  $X$ . A *prime divisor* on  $X$  is an irreducible hypersurface  $D \subseteq X$ . The group of *Weil divisors* on  $X$  is the free abelian group  $\text{WDiv}(X)$  generated by the prime divisors. We write  $D \geq 0$  for a Weil divisor  $D$ , if it is a nonnegative linear combination of prime divisors. To every rational function  $f \in \mathbb{K}(X)^*$  one associates its *principal divisor*

$$\text{div}(f) := \sum_D \text{ord}_D(f) D \in \text{WDiv}(X),$$

where  $D$  runs through the prime divisors of  $X$  and  $\text{ord}_D(f)$  is the vanishing order of  $f$  at  $D$ . The *divisor class group*  $\text{Cl}(X)$  of  $X$  is the factor group of  $\text{WDiv}(X)$  by the subgroup  $\text{PDiv}(X)$  of principal divisors. To every Weil divisor  $D$  on  $X$ , one associates a *sheaf*  $\mathcal{O}_X(D)$  of  $\mathcal{O}_X$ -modules: for any open  $U \subseteq X$  one sets

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in \mathbb{K}(X)^*; (\text{div}(f) + D)|_U \geq 0\} \cup \{0\},$$

where the restriction map  $\text{WDiv}(X) \rightarrow \text{WDiv}(U)$  is defined for a prime divisor  $D$  as  $D|_U := D \cap U$  if it intersects  $U$  and  $D|_U := 0$  otherwise. The sheaf  $\mathcal{O}_X(D)$  is reflexive, i.e. canonically isomorphic to its double dual, and of rank one. Note that for any two functions  $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$  and  $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$  the product  $f_1 f_2$  belongs to  $\Gamma(U, \mathcal{O}_X(D_1 + D_2))$ .

**Definition 1.1.** The *sheaf of divisorial algebras* associated to a subgroup  $K \subseteq \text{WDiv}(X)$  is the sheaf of  $K$ -graded  $\mathcal{O}_X$ -algebras

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{S}_D, \quad \mathcal{S}_D := \mathcal{O}_X(D),$$

where the multiplication in  $\mathcal{S}$  is defined by multiplying homogeneous sections in the field of functions  $\mathbb{K}(X)$ .

The sheaf of divisorial algebras associated to a finitely generated group of Weil divisors turns out to be a sheaf of normal algebras. A crucial observation is the following.

**Proposition 1.2.** *Let  $\mathcal{S}$  be the sheaf of divisorial algebras associated to a finitely generated group  $K \subseteq \text{WDiv}(X)$ . If the canonical map  $K \rightarrow \text{Cl}(X)$  sending  $D \in K$  to its class  $[D] \in \text{Cl}(X)$  is surjective, then  $\Gamma(X, \mathcal{S})$  is a unique factorization domain.*

**Example 1.3.** On the projective line  $X = \mathbb{P}_1$ , consider  $D := \{\infty\}$ , the group  $K := \mathbb{Z}D$ , and the associated  $K$ -graded sheaf of algebras  $\mathcal{S}$ . Then we have isomorphisms

$$\varphi_n: \mathbb{K}[T_0, T_1]_n \rightarrow \Gamma(\mathbb{P}_1, \mathcal{S}_{nD}), \quad f \mapsto f(1, z),$$

where  $\mathbb{K}[T_0, T_1]_n \subseteq \mathbb{K}[T_0, T_1]$  denotes the vector space of all polynomials homogeneous of degree  $n$ . Putting them together we obtain a graded isomorphism

$$\mathbb{K}[T_0, T_1] \cong \Gamma(\mathbb{P}_1, \mathcal{S}).$$

**Example 1.4** (The affine quadric surface). Consider the two-dimensional affine variety

$$X := V(\mathbb{K}^3; T_1T_2 - T_3^2) \subseteq \mathbb{K}^3.$$

We have the functions  $f_i := T_{i|X}$  on  $X$  and with the prime divisors  $D_1 := V(X; f_1)$  and  $D_2 := V(X; f_2)$  on  $X$ , we have

$$\text{div}(f_1) = 2D_1, \quad \text{div}(f_2) = 2D_2, \quad \text{div}(f_3) = D_1 + D_2.$$

For  $K := \mathbb{Z}D_1$ , let  $\mathcal{S}$  denote the associated sheaf of divisorial algebras. Consider the sections

$$\begin{aligned} g_1 &:= 1 \in \Gamma(X, \mathcal{S}_{D_1}), & g_2 &:= f_3 f_1^{-1} \in \Gamma(X, \mathcal{S}_{D_1}), \\ g_3 &:= f_1^{-1} \in \Gamma(X, \mathcal{S}_{2D_1}), & g_4 &:= f_1 \in \Gamma(X, \mathcal{S}_{-2D_1}). \end{aligned}$$

Then  $g_1, g_2$  generate  $\Gamma(X, \mathcal{S}_{D_1})$  as a  $\Gamma(X, \mathcal{S}_0)$ -module, and  $g_3, g_4$  are inverse to each other. Moreover, we have

$$f_1 = g_1^2 g_4, \quad f_2 = g_2^2 g_4, \quad f_3 = g_1 g_2 g_4.$$

Thus,  $g_1, g_2, g_3$  and  $g_4$  generate the  $\mathbb{K}$ -algebra  $\Gamma(X, \mathcal{S})$ . Setting  $\deg(Z_i) := \deg(g_i)$ , we obtain a  $K$ -graded isomorphism

$$\mathbb{K}[Z_1, Z_2, Z_3^{\pm 1}] \rightarrow \Gamma(X, \mathcal{S}), \quad Z_1 \mapsto g_1, \quad Z_2 \mapsto g_2, \quad Z_3 \mapsto g_3.$$

We are ready to define the Cox ring. From now on  $X$  is a normal variety with  $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$  and finitely generated divisor class group  $\text{Cl}(X)$ . The idea is to start with the sheaf  $\mathcal{S}$  of divisorial algebras associated to a group  $K$  of Weil divisors projecting onto  $\text{Cl}(X)$  and then to identify systematically isomorphic homogeneous components  $\mathcal{S}_D$  and  $\mathcal{S}'_D$  via dividing by a suitable sheaf of ideals.

**Construction 1.5.** Fix a subgroup  $K \subseteq \text{WDiv}(X)$  such that the map  $c: K \rightarrow \text{Cl}(X)$  sending  $D \in K$  to its class  $[D] \in \text{Cl}(X)$  is surjective. Let  $K^0 \subseteq K$  be the kernel of  $c$ , and let  $\chi: K^0 \rightarrow \mathbb{K}(X)^*$  be a character, i.e. a group homomorphism, with

$$\text{div}(\chi(E)) = E, \quad \text{for all } E \in K^0.$$

Let  $\mathcal{S}$  be the sheaf of divisorial algebras associated to  $K$  and denote by  $\mathcal{I}$  the sheaf of ideals of  $\mathcal{S}$  locally generated by the sections  $1 - \chi(E)$ , where  $1$  is homogeneous of degree zero,  $E$  runs through  $K^0$  and  $\chi(E)$  is homogeneous of degree  $-E$ . The *Cox sheaf* associated to  $K$  and  $\chi$  is the quotient sheaf  $\mathcal{R} := \mathcal{S}/\mathcal{I}$  together with the  $\text{Cl}(X)$ -grading

$$\mathcal{R} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \pi \left( \bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'} \right).$$

where  $\pi: \mathcal{S} \rightarrow \mathcal{R}$  denotes the projection. The Cox sheaf  $\mathcal{R}$  is a quasicoherent sheaf of  $\text{Cl}(X)$ -graded  $\mathcal{O}_X$ -algebras. The *Cox ring* is the ring of global sections

$$\mathcal{R}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{R}_{[D]}(X), \quad \mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{R}_{[D]}).$$

In general, the Cox sheaf is not a sheaf of divisorial algebras. However, the following shows that it is not too far from them.

**Remark 1.6.** Situation as in Construction 1.5. Then, for every  $D \in K$ , we have an isomorphism of sheaves

$$\pi_{|\mathcal{S}_D}: \mathcal{S}_D \rightarrow \mathcal{R}_{[D]}.$$

Moreover, for every open subset  $U \subseteq X$ , we have a canonical isomorphism on the level of sections

$$\Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I}) \cong \Gamma(U, \mathcal{S}/\mathcal{I}).$$

In particular, the Cox ring  $\mathcal{R}(X)$  is isomorphic to the quotient  $\Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I})$  of global sections.

**Proposition 1.7.** *If  $K, \chi$  and  $K', \chi'$  are data as in Construction 1.5, then the associated Cox sheaves are isomorphic as graded sheaves.*

**Example 1.8** (The affine quadric surface, continued). Look again at the two-dimensional affine variety discussed in 1.4:

$$X := V(\mathbb{K}^3; T_1T_2 - T_3^2) \subseteq \mathbb{K}^3.$$

The divisor class group  $\text{Cl}(X)$  is of order two; it is generated by  $[D_1]$ . The kernel of the projection  $K \rightarrow \text{Cl}(X)$  is  $K^0 = 2\mathbb{Z}D_1$  and a character as in Construction 1.5 is

$$\chi: K^0 \rightarrow \mathbb{K}(X)^*, \quad 2nD_1 \mapsto f_1^n.$$

The ideal  $\mathcal{I}$  is globally generated by the section  $1 - f_1$ , where  $f_1 \in \Gamma(X, \mathcal{S}_{-2D_1})$ . Consequently, the Cox ring of  $X$  is given as

$$\mathcal{R}(X) \cong \Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I}) \cong \mathbb{K}[Z_1, Z_2, Z_3^{\pm 1}]/\langle 1 - Z_3^{-1} \rangle \cong \mathbb{K}[Z_1, Z_2],$$

where the  $\text{Cl}(X)$ -grading on the polynomial ring  $\mathbb{K}[Z_1, Z_2]$  is given by  $\deg(Z_1) = \deg(Z_2) = [D_1]$ .

The construction of Cox sheaves (and thus also Cox rings) of a variety can be made canonical by fixing a suitable point. We say that  $x \in X$  is *factorial*, if the local ring  $\mathcal{O}_{X,x}$  is factorial; that means that every Weil divisor is principal near  $x$ .

**Construction 1.9.** Let  $x \in X$  be a factorial point and consider the sheaf of divisorial algebras  $\mathcal{S}^x$  associated to the group of Weil divisors avoiding  $x$ :

$$K^x := \{D \in \text{WDiv}(X); x \notin \text{Supp}(D)\} \subseteq \text{WDiv}(X).$$

Let  $K^{x,0} \subseteq K^x$  be the subgroup consisting of principal divisors. Then, for every  $E \in K^{x,0}$ , there is a unique  $f_E \in \Gamma(X, \mathcal{S}_{-E})$ , which is defined near  $x$  and satisfies

$$\text{div}(f_E) = E, \quad f_E(x) = 1.$$

The map  $\chi^x: K^x \rightarrow \mathbb{K}(X)^*$  sending  $E$  to  $f_E$  is a character as in 1.5. We call the resulting Cox sheaf  $\mathcal{R}^x$  the *canonical Cox sheaf of the pointed space*  $(X, x)$ .

As noted in 1.6, the homogeneous components of the Cox sheaf are isomorphic to reflexive rank one sheaves  $\mathcal{O}_X(D)$ . The following associates to every homogeneous section of the Cox ring a divisor, not depending on the choices made in 1.5.

**Construction 1.10.** In the setting of Construction 1.5, let  $D \in K$  and consider an element  $0 \neq f \in \mathcal{R}_{[D]}(X)$ . By Remark 1.6, there is a (unique)  $\tilde{f} \in \Gamma(X, \mathcal{S}_D)$  with  $\pi(\tilde{f}) = f$ . The  $[D]$ -divisor and the  $[D]$ -localization of  $f$  are

$$\text{div}_{[D]}(f) := \text{div}(\tilde{f}) + D, \quad X_{[D],f} := X \setminus \text{Supp}(\text{div}_{[D]}(f)).$$

We now discuss basic algebraic properties of the Cox ring. Recall that a *Krull ring* is an integral ring  $R$  with a family  $(\nu_i)_{i \in I}$  of discrete valuations such that for every non-zero  $f$  in the quotient field  $Q(R)$ , one has  $\nu_i(f) \neq 0$  only for finitely many  $i \in I$  and  $f \in R$  if and only if  $\nu_i(f) \geq 0$  holds for all  $i \in I$ .

**Theorem 1.11.** *The Cox ring  $\mathcal{R}(X)$  is an (integral) normal Krull ring. Moreover, one has the following statements on localization and units.*

- (i) *For every non-zero homogeneous  $f \in \mathcal{R}_{[D]}(X)$ , there is a canonical isomorphism*

$$\Gamma(X, \mathcal{R})_f \cong \Gamma(X_{[D],f}, \mathcal{R}).$$

- (ii) *Every homogeneous unit of  $\mathcal{R}(X)$  is constant. If  $X$  is complete then even  $\mathcal{R}(X)^* = \mathbb{K}^*$  holds.*

Finally, we take a closer look at divisibility properties of Cox rings. In general, they are not factorial, i.e. unique factorization domains, but the following weaker property will be guaranteed.

**Definition 1.12.** Consider an abelian group  $K$  and a  $K$ -graded integral  $\mathbb{K}$ -algebra  $R = \bigoplus_K R_w$ .

- (i) A non-zero non-unit  $f \in R$  is  $K$ -prime if it is homogeneous and  $f|gh$  with homogeneous  $g, h \in R$  implies  $f|g$  or  $f|h$ .
- (ii) We say that  $R$  is *factorially graded* if every homogeneous non-zero non-unit  $f \in R$  is a product of  $K$ -primes.

**Remark 1.13.** The concepts “factorially graded” and “factorial” coincide if the grading group is torsion free. As soon as we have torsion in the grading group, they may differ. For example, consider

$$R := \mathbb{K}[T_1, T_2, T_3] / \langle T_1^2 + T_2^2 + T_3^2 \rangle.$$

This is surely not a factorial ring. But it becomes factorially graded by the group  $K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  when we set

$$\deg(T_1) := (1, \bar{0}, \bar{0}), \quad \deg(T_2) := (1, \bar{1}, \bar{0}), \quad \deg(T_3) := (1, \bar{0}, \bar{1}).$$

**Theorem 1.14.** *Suppose that the Cox ring  $\mathcal{R}(X)$  satisfies  $\mathcal{R}(X)^* = \mathbb{K}^*$ ; for example, assume  $X$  to be complete. Then  $\mathcal{R}(X)$  is factorially  $\text{Cl}(X)$ -graded. If moreover  $\text{Cl}(X)$  is torsion free, then  $\mathcal{R}(X)$  is factorial.*

The following statement shows that divisibility in the Cox ring  $\mathcal{R}(X)$  can be formulated geometrically in terms of  $[D]$ -divisors on the underlying variety  $X$ .

**Proposition 1.15.** *Suppose that the Cox ring  $\mathcal{R}(X)$  satisfies  $\mathcal{R}(X)^* = \mathbb{K}^*$ .*

- (i) *An element  $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$  divides  $0 \neq g \in \Gamma(X, \mathcal{R}_{[E]})$  if and only if  $\text{div}_{[D]}(f) \leq \text{div}_{[E]}(g)$  holds.*
- (ii) *An element  $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$  is  $\text{Cl}(X)$ -prime if and only if the divisor  $\text{div}_{[D]}(f) \in \text{WDiv}(X)$  is prime.*

**1.2. Quasitorus actions.** Here we recall the correspondence between affine algebras  $A$  graded by a finitely generated abelian group  $K$  and affine varieties  $X$  coming with an action of a quasitorus  $H$ . Moreover, we discuss good quotients and their basic properties. Standard references are [8, 35, 39, 28, 31].

An *affine algebraic group* is an affine variety  $G$  together with a group structure such that the group operations are morphisms. A *morphism of affine algebraic groups* is a morphism of the underlying varieties which is moreover a group homomorphism. A *character* is a morphism  $G \rightarrow \mathbb{K}^*$  to the multiplicative group of the ground field. Endowed with pointwise multiplication, the characters of  $G$  form an abelian group  $\mathbb{X}(G)$ .

**Definition 1.16.** A *quasitorus*, also called a *diagonalizable group*, is an affine algebraic group  $H$  whose algebra of regular functions  $\Gamma(H, \mathcal{O})$  is generated as a  $\mathbb{K}$ -vector space by the characters  $\chi \in \mathbb{X}(H)$ . A *torus* is a connected quasitorus.

**Example 1.17.** The *standard  $n$ -torus* is  $\mathbb{T}^n := (\mathbb{K}^*)^n$ . Its characters are precisely the Laurent monomials  $T^\nu = T_1^{\nu_1} \cdots T_n^{\nu_n}$ , where  $\nu \in \mathbb{Z}^n$ , and its algebra of regular functions is the Laurent polynomial algebra

$$\Gamma(\mathbb{T}^n, \mathcal{O}) = \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}] = \bigoplus_{\nu \in \mathbb{Z}^n} \mathbb{K} \cdot T^\nu = \mathbb{K}[\mathbb{Z}^n].$$

To any finitely generated abelian group  $K$  one associates in a functorial way a quasitorus, namely  $H := \text{Spec } \mathbb{K}[K]$ , the spectrum of the group algebra  $\mathbb{K}[K]$ .

**Remark 1.18.** The quasitorus  $H := \text{Spec } \mathbb{K}[K]$  can be realized as a closed subgroup of a standard  $r$ -torus as follows. By the elementary divisors theorem, we find generators  $w_1, \dots, w_r$  of  $K$  such that the epimorphism  $\mathbb{Z}^r \rightarrow K$ ,  $e_i \mapsto w_i$  has kernel

$$\mathbb{Z}a_1e_1 \oplus \dots \oplus \mathbb{Z}a_se_s \subseteq \mathbb{Z}^r, \quad a_1, \dots, a_s \in \mathbb{Z}_{\geq 1}.$$

The corresponding morphism  $H \rightarrow \mathbb{T}^r$  is a closed embedding realizing  $H$  as the kernel of  $\mathbb{T}^r \rightarrow \mathbb{T}^s$ ,  $(t_1, \dots, t_r) \mapsto (t_1^{a_1}, \dots, t_s^{a_s})$ . In particular, we see that  $H$  is a direct product of a torus and a finite abelian group:

$$H \cong C(a_1) \times \dots \times C(a_s) \times \mathbb{T}^{r-s}, \quad C(a_i) := \{\zeta \in \mathbb{K}^*; \zeta^{a_i} = 1\}.$$

**Theorem 1.19.** *We have contravariant exact functors being essentially inverse to each other:*

$$\begin{aligned} \{\text{finitely generated abelian groups}\} &\longleftrightarrow \{\text{quasitori}\} \\ K &\mapsto \text{Spec } \mathbb{K}[K], \\ \mathbb{X}(H) &\leftarrow H. \end{aligned}$$

*Under these equivalences, the free finitely generated abelian groups correspond to the tori.*

Quasitori are linearly reductive in characteristic zero: every rational representation even splits into one-dimensional ones. Applying this to the representation on the algebra of global functions of an affine variety with quasitorus action, we obtain a grading by the character group in the following way.

**Remark 1.20.** Let a quasitorus  $H$  act on a not necessarily affine variety  $X$ . Then the algebra  $\Gamma(X, \mathcal{O})$  becomes  $\mathbb{X}(H)$ -graded via

$$\Gamma(X, \mathcal{O}) = \bigoplus_{\chi \in \mathbb{X}(H)} \Gamma(X, \mathcal{O})_\chi, \quad \Gamma(X, \mathcal{O})_\chi := \{f \in \Gamma(X, \mathcal{O}); f(h \cdot x) = \chi(h)f(x)\}.$$

We now associate in functorial manner to every affine algebra  $A = \bigoplus_K A_w$  graded by a finitely generated abelian group  $K$  the affine variety  $X = \text{Spec } A$  with an action of the quasitorus  $H = \text{Spec } \mathbb{K}[K]$ . Again this can be made concrete.

**Construction 1.21.** Let  $K$  be a finitely generated abelian group and  $A$  a  $K$ -graded affine algebra. Set  $X = \text{Spec } A$ . If  $f_i \in A_{w_i}$ ,  $i = 1, \dots, r$ , generate  $A$ , then we have a closed embedding

$$X \rightarrow \mathbb{K}^r, \quad x \mapsto (f_1(x), \dots, f_r(x)),$$

and  $X \subseteq \mathbb{K}^r$  is invariant under the diagonal action of  $H = \text{Spec } \mathbb{K}[K]$  given by the characters  $\chi^{w_1}, \dots, \chi^{w_r}$ . Note that for any  $f \in A$  homogeneity is characterized by

$$f \in A_w \iff f(h \cdot x) = \chi^w(h)f(x) \text{ for all } h \in H, x \in X.$$

**Theorem 1.22.** *We have contravariant functors being essentially inverse to each other:*

$$\begin{aligned} \{\text{graded affine algebras}\} &\longleftrightarrow \{\text{affine varieties with quasitorus action}\} \\ A &\mapsto \text{Spec } A, \\ \Gamma(X, \mathcal{O}) &\leftarrow X. \end{aligned}$$

**Definition 1.23.** Let  $G$  be a linearly reductive affine algebraic group  $G$  acting on a variety  $X$ . A morphism  $p: X \rightarrow Y$  is called a *good quotient* for this action if it has the following properties:

- (i)  $p: X \rightarrow Y$  is affine and  $G$ -invariant,
- (ii) the pullback  $p^*: \mathcal{O}_Y \rightarrow (p_*\mathcal{O}_X)^G$  is an isomorphism.

A morphism  $p: X \rightarrow Y$  is called a *geometric quotient* if it is a good quotient and its fibers are precisely the  $G$ -orbits.

**Remark 1.24.** Let a linearly reductive group  $G$  act on an affine variety  $X$ . Then Hilbert's finiteness theorem ensures that the algebra of invariants  $\Gamma(X, \mathcal{O})^G$  is finitely generated. This gives a good quotient  $p: X \rightarrow Y$  with  $Y := \text{Spec } \Gamma(X, \mathcal{O})^G$ .

**Remark 1.25.** Let  $A = \oplus_K A_w$  be an affine algebra graded by a finitely generated abelian group  $K$ . Then  $H = \text{Spec } \mathbb{K}[K]$  acts on  $X = \text{Spec } A$  and we have

$$\Gamma(X, \mathcal{O})^H = A_0.$$

In order to compute  $A_0$ , choose homogeneous generators  $f_1, \dots, f_r$  of  $A$  and consider the homomorphism

$$Q: \mathbb{Z}^r \rightarrow K, \quad e_i \mapsto \deg(f_i).$$

Then, for any set  $B$  of generators of the monoid  $\mathbb{Z}_{\geq 0}^r \cap \ker(Q)$ , the algebra  $A_0$  of invariants is generated by the products  $f_1^{\nu_1} \cdots f_r^{\nu_r}$ , where  $\nu \in B$ .

The basic properties of good quotients are that they map closed invariant subsets to closed sets and that they separate disjoint closed invariant sets. An immediate consequence is that the target space carries the quotient topology. Another application is the following statement on the fibers.

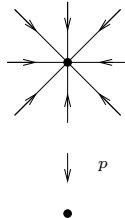
**Proposition 1.26.** *Let a linearly reductive algebraic group  $G$  act on a variety  $X$ , and let  $p: X \rightarrow Y$  be a good quotient. Then  $p$  is surjective and for any  $y \in Y$  one has:*

- (i) *There is exactly one closed  $G$ -orbit  $G \cdot x$  in the fiber  $p^{-1}(y)$ .*
- (ii) *Every orbit  $G \cdot x' \subseteq p^{-1}(y)$  has  $G \cdot x$  in its closure.*

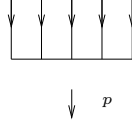
The first statement means that a good quotient  $p: X \rightarrow Y$  parametrizes the closed orbits of the  $G$ -variety  $X$ . Using the description of the fibers one easily verifies that a good quotient is categorical, i.e. universal with respect to invariant morphisms. In particular, the quotient space is unique up to isomorphism which justifies the common notation  $X//G$ .

**Example 1.27.** Consider the  $\mathbb{K}^*$ -action  $t \cdot (z_1, z_2) = (t^a z_1, t^b z_2)$  on  $\mathbb{K}^2$ . The following three cases are typical.

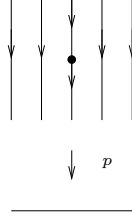
- (i) We have  $a = b = 1$ . Every  $\mathbb{K}^*$ -invariant function is constant and the constant map  $p: \mathbb{K}^2 \rightarrow \{\text{pt}\}$  is a good quotient.



- (ii) We have  $a = 0$  and  $b = 1$ . The algebra of  $\mathbb{K}^*$ -invariant functions is generated by  $z_1$  and the map  $p: \mathbb{K}^2 \rightarrow \mathbb{K}, (z_1, z_2) \mapsto z_1$  is a good quotient.



- (iii) We have  $a = 1$  and  $b = -1$ . The algebra of  $\mathbb{K}^*$ -invariant functions is generated by  $z_1 z_2$  and  $p: \mathbb{K}^2 \rightarrow \mathbb{K}$ ,  $(z_1, z_2) \mapsto z_1 z_2$  is a good quotient.



Note that the general  $p$ -fiber is a single  $\mathbb{K}^*$ -orbit, whereas  $p^{-1}(0)$  consists of three orbits and is reducible.

**Example 1.28** (The affine quadric surface as a quotient). Consider the action of the multiplicative group  $C(2) = \{1, -1\}$  on  $\mathbb{K}^2$  given by

$$\zeta \cdot z := (\zeta z_1, \zeta z_2).$$

This action has 0 as a fixed point and any  $z \neq 0$  has trivial isotropy group. The algebra  $A \subseteq \mathbb{K}^2$  of invariants is generated by

$$f_{11} := T_1^2, \quad f_{22} := T_2^2, \quad f_{12} := T_1 T_2.$$

The ideal of relations among them is generated by the polynomial  $f_{11} f_{22} - f_{12}^2$ . Consequently, with  $X := V(w_1 w_2 - w_3^2) \subseteq \mathbb{K}^3$ , the quotient map is

$$\pi: \mathbb{K}^3 \rightarrow X, \quad z \mapsto (f_{11}(z), f_{22}(z), f_{12}(z)).$$

The fibers of  $\pi$  are precisely the  $C(2)$ -orbits. In particular,  $\pi$  is a geometric quotient (as it holds for any finite group action on an affine variety).

**Example 1.29** (The affine quadric threefold as a quotient). Consider the action of  $\mathbb{K}^*$  on  $\mathbb{K}^4$  given by

$$t \cdot z := (t z_1, t^{-1} z_2, t z_3, t^{-1} z_4).$$

Note that  $\mathbb{K}^*$  has 0 as a fixed point and any  $z \neq 0$  has trivial isotropy group. The algebra  $A \subseteq \mathbb{K}[T_1, \dots, T_4]$  of invariants is generated by

$$f_{12} := T_1 T_2, \quad f_{34} := T_3 T_4, \quad f_{14} := T_1 T_4, \quad f_{23} := T_2 T_3.$$

The ideal of relations is generated by the polynomial  $f_{12} f_{34} - f_{14} f_{23}$ . Thus, with  $X := V(w_1 w_2 - w_3 w_4) \subseteq \mathbb{K}^4$ , the quotient map is

$$\pi: \mathbb{K}^4 \rightarrow X, \quad z \mapsto (f_{12}(z), f_{34}(z), f_{14}(z), f_{23}(z)).$$

The fiber over any point  $0 \neq x$  is a free  $\mathbb{K}^*$ -orbit and hence isomorphic to  $\mathbb{K}^*$ , whereas the fiber over the point  $0 \in X$  is given by

$$\pi^{-1}(0) = V(T_1, T_3) \cup V(T_2, T_4) \subseteq \mathbb{K}^4.$$

**1.3. Characteristic spaces.** As before,  $X$  is a normal variety with  $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ . We discuss the geometric counterpart of a Cox sheaf  $\mathcal{R}$  on  $X$ , its relative spectrum. In order to obtain a reasonable object,  $\mathcal{R}$  should be locally of finite type. By Theorem 1.11, this holds, if the Cox ring  $\mathcal{R}(X)$  is finitely generated. Moreover, one can show that  $\mathcal{R}$  is locally of finite type for every  $\mathbb{Q}$ -factorial  $X$ .



**Construction 1.30.** Let  $\mathcal{R}$  be a Cox sheaf on  $X$  and suppose that  $\mathcal{R}$  is locally of finite type. Then the relative spectrum

$$\widehat{X} := \operatorname{Spec}_X(\mathcal{R})$$

is a quas affine variety. The  $\operatorname{Cl}(X)$ -grading of the sheaf  $\mathcal{R}$  defines an action of the diagonalizable group

$$H_X := \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$$

on  $\widehat{X}$ . The canonical morphism  $q_X: \widehat{X} \rightarrow X$  is a good quotient for this action, and we have an isomorphism of graded sheaves

$$\mathcal{R} \cong (q_X)_*(\mathcal{O}_{\widehat{X}}).$$

We call  $q_X: \widehat{X} \rightarrow X$  the *characteristic space* associated to  $\mathcal{R}$ , and  $H_X$  the *characteristic quasitorus* of  $X$ .

In the case of a locally factorial variety  $X$ , the characteristic space coincides with the universal torsor over  $X$ . As soon as  $X$  has non-factorial singularities the two concepts differ from each other, as we will indicate below.

**Proposition 1.31.** *Consider the characteristic space  $q_X: \widehat{X} \rightarrow X$ .*

- (i) *The inverse image  $q_X^{-1}(X_{\text{reg}})$  of the set of smooth points is smooth,  $H_X$  acts freely there and  $q_X^{-1}(X_{\text{reg}}) \rightarrow X_{\text{reg}}$  is an étale  $H_X$ -principal bundle.*
- (ii) *For any closed set  $A \subseteq X$  of codimension at least two, the inverse image  $q_X^{-1}(A) \subseteq \widehat{X}$  is as well of codimension at least two.*
- (iii) *Let  $\widehat{x} \in \widehat{X}$  be a point such that the orbit  $H_X \cdot \widehat{x} \subseteq \widehat{X}$  is closed, and consider an element  $f \in \Gamma(X, \mathcal{R}_{[D]})$ . Then we have*

$$f(\widehat{x}) = 0 \iff q_X(\widehat{x}) \in \operatorname{Supp}(\operatorname{div}_{[D]}(f)).$$

We now relate properties of the  $H_X$ -action to geometric properties on  $X$ . For  $x \in X$ , let  $\operatorname{PDiv}(X, x) \subseteq \operatorname{WDiv}(X)$  denote the subgroup of all Weil divisors, which are principal on some neighbourhood of  $x$ . We define the *local class group* of  $X$  at  $x$  to be the factor group

$$\operatorname{Cl}(X, x) := \operatorname{WDiv}(X) / \operatorname{PDiv}(X, x).$$

Obviously the group  $\operatorname{PDiv}(X)$  of principal divisors is contained in  $\operatorname{PDiv}(X, x)$ . Thus, there is a canonical epimorphism  $\pi_x: \operatorname{Cl}(X) \rightarrow \operatorname{Cl}(X, x)$ . We denote by  $H_{X, \widehat{x}} \subseteq H_X$  the isotropy group of  $\widehat{x} \in \widehat{X}$ .

**Proposition 1.32.** *Consider the characteristic space  $q_X: \widehat{X} \rightarrow X$ . Given  $x \in X$ , fix a point  $\widehat{x} \in q_X^{-1}(x)$  with closed  $H_X$ -orbit. Then we have a canonical isomorphism*

$$\operatorname{Cl}(X, x) \cong \mathbb{X}(H_{X, \widehat{x}}).$$

Recall that a point  $x \in X$  is factorial if and only if near  $x$  every Weil divisor is principal. We say that  $x \in X$  is  $\mathbb{Q}$ -factorial if near  $x$  for every Weil divisor some multiple is principal.

**Corollary 1.33.** *Consider the characteristic space  $q_X: \widehat{X} \rightarrow X$ .*

- (i) *A point  $x \in X$  is factorial if and only if the fiber  $q_X^{-1}(x)$  is a single  $H_X$ -orbit with trivial isotropy.*
- (ii) *A point  $x \in X$  is  $\mathbb{Q}$ -factorial if and only if the fiber  $q_X^{-1}(x)$  is a single  $H_X$ -orbit.*

The first part of the following says in particular that characteristic space and universal torsor coincide if and only if  $X$  has at most factorial singularities.

**Corollary 1.34.** *Consider the characteristic space  $q_X: \widehat{X} \rightarrow X$ .*

- (i) *The action of  $H_X$  on  $\widehat{X}$  is free if and only if  $X$  is locally factorial.*

(ii) The good quotient  $q_X: \hat{X} \rightarrow X$  is geometric if and only if  $X$  is  $\mathbb{Q}$ -factorial.

Recall that the *Picard group* of  $X$  is the factor group of the group  $\text{CDiv}(X)$  of locally principal Weil divisors by the subgroup of principal divisors:

$$\text{Pic}(X) = \text{CDiv}(X)/\text{PDiv}(X) = \bigcap_{x \in X} \ker(\pi_x).$$

**Corollary 1.35.** Consider the characteristic space  $q_X: \hat{X} \rightarrow X$ . Let  $\hat{H}_X \subseteq H_X$  be the subgroup generated by all isotropy groups  $H_{X,\hat{x}}$ , where  $\hat{x} \in \hat{X}$ . Then we have

$$\ker(\mathbb{X}(H_X) \rightarrow \mathbb{X}(\hat{H}_X)) = \bigcap_{\hat{x} \in \hat{X}} \ker(\mathbb{X}(H_X) \rightarrow \mathbb{X}(H_{X,\hat{x}}))$$

and the projection  $H_X \rightarrow H_X/\hat{H}_X$  corresponds to the inclusion  $\text{Pic}(X) \subseteq \text{Cl}(X)$  of character groups.

**Corollary 1.36.** If the variety  $\hat{X}$  contains an  $H_X$ -fixed point, then the Picard group  $\text{Pic}(X)$  is trivial.

As we noted, the characteristic space of a variety  $X$  is a quas affine variety  $\hat{X}$  with an action of the characteristic quasitorus  $H_X$  having  $X$  as a good quotient. Our next aim is to characterize this situation in terms of Geometric Invariant Theory.

**Definition 1.37.** Let  $G$  be an affine algebraic group and  $W$  a  $G$ -variety. We say that the  $G$ -action on  $W$  is *strongly stable* if there is an open invariant subset  $W' \subseteq W$  with the following properties:

- (i) the complement  $W \setminus W'$  is of codimension at least two in  $W$ ,
- (ii) the group  $G$  acts freely, i.e. with trivial isotropy groups, on  $W'$ ,
- (iii) for every  $x \in W'$  the orbit  $G \cdot x$  is closed in  $W$ .

**Definition 1.38.** Let a group  $G$  act on a normal variety  $Y$ . A divisor  $\sum a_D D$  is called  *$G$ -invariant* if its multiplicities satisfy  $a_{gD} = a_D$  for every  $g \in G$ . We say that  $Y$  is  *$G$ -factorial* if every  $G$ -invariant divisor is principal.

Note that a quas affine variety with a quasitorus  $H$  acting on it is  *$H$ -factorial* if and only if its ring of functions is factorially graded by the character group  $\mathbb{X}(H)$ .

**Remark 1.39.** Let  $X$  be a normal variety with characteristic space  $q_X: \hat{X} \rightarrow X$ . Then  $\hat{X}$  is  $H_X$ -factorial and  $q_X^{-1}(X_{\text{reg}}) \subseteq \hat{X}$  satisfies the required properties of  $W' \subseteq W$  of Definition 1.37.

**Theorem 1.40.** Let a quasitorus  $H$  act on a normal quas affine variety  $\mathcal{X}$  with a good quotient  $q: \mathcal{X} \rightarrow X$ . Assume that  $\Gamma(\mathcal{X}, \mathcal{O}^*) = \mathbb{K}^*$  holds,  $\mathcal{X}$  is  $H$ -factorial and the  $H$ -action is strongly stable. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow[\cong]{\mu} & \hat{X} \\ & \searrow q & \swarrow q_X \\ & X & \end{array}$$

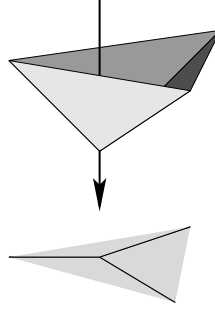
where the quotient space  $X$  is a normal variety with  $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$ , we have  $\text{Cl}(X) = \mathbb{X}(H)$  and  $q_X: \hat{X} \rightarrow X$  is a characteristic space for  $X$  and the isomorphism  $\mu: \mathcal{X} \rightarrow \hat{X}$  is equivariant with respect to the actions of  $H = H_X$ .

**Example 1.41** (The affine quadric surface, continued). In 1.28, we realized the surface  $X = V(T_1 T_2 - T_3^2)$  as a strongly stable quotient of  $\mathbb{K}^2$  by  $\mathbb{Z}/2\mathbb{Z}$ . Thus,  $\text{Cl}(X)$  is of order two and  $\mathbb{K}^2 \rightarrow X$  is a characteristic space. Moreover, from 1.36 we infer  $\text{Pic}(X) = 0$ .

**Example 1.42** (The affine quadric threefold, continued). In 1.29, we realized the affine threefold  $X = V(T_1T_2 - T_3T_4)$  as a strongly stable quotient of  $\mathbb{K}^4$  by  $\mathbb{K}^*$ . Thus, we have  $\text{Cl}(X) \cong \mathbb{Z}$  and  $\mathbb{K}^4 \rightarrow X$  is a characteristic space. Moreover, from 1.36 we infer  $\text{Pic}(X) = 0$ .

These two examples are special cases of the much bigger class of *toric varieties*, i.e., normal varieties  $X$  endowed with a torus action  $T \times X \rightarrow X$  such that for some  $x_0 \in X$  the orbit map  $T \rightarrow X$ ,  $t \mapsto t \cdot x_0$  is an open embedding. Toric varieties admit a complete description in terms of lattice fans; standard references are [15, 33, 18, 14]. In this picture, Cox ring and characteristic space look as follows; see [13, 6, 7, 32].

**Construction 1.43.** Assume that the toric variety  $X$  arises from a fan  $\Sigma$  in a lattice  $N$ . The condition  $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$  means that the primitive vectors  $v_1, \dots, v_r \in N$  on the rays of  $\Sigma$  generate  $N_{\mathbb{Q}}$  as a vector space.



Set  $F := \mathbb{Z}^r$  and consider the linear map  $P: F \rightarrow N$  sending the  $i$ -th canonical base vector  $f_i \in F$  to  $v_i \in N$ . There is a fan  $\widehat{\Sigma}$  in  $F$  consisting of certain faces of the positive orthant  $\delta \subseteq F_{\mathbb{Q}}$ , namely

$$\widehat{\Sigma} := \{\widehat{\sigma} \preceq \delta; P(\widehat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The fan  $\widehat{\Sigma}$  defines an open toric subvariety  $\widehat{X}$  of  $\overline{X} = \text{Spec}(\mathbb{K}[\delta^{\vee} \cap E])$ , where  $E := \text{Hom}(F, \mathbb{Z})$ . Note that all rays  $\text{cone}(f_1), \dots, \text{cone}(f_r)$  of the positive orthant  $\delta \subseteq F_{\mathbb{Q}}$  belong to  $\widehat{\Sigma}$  and thus we have

$$\Gamma(\widehat{X}, \mathcal{O}) = \Gamma(\overline{X}, \mathcal{O}) = \mathbb{K}[\delta^{\vee} \cap E].$$

As  $P: F \rightarrow N$  is a map of the fans  $\widehat{\Sigma}$  and  $\Sigma$ , i.e., sends cones of  $\widehat{\Sigma}$  into cones of  $\Sigma$ , it defines a morphism  $p: \widehat{X} \rightarrow X$  of toric varieties. Note that we have  $\overline{X} = \mathbb{K}^r$  and in terms of the coordinates  $T_i = \chi^{e_i}$ , the open subset  $\widehat{X} \subseteq \overline{X}$  is given as

$$\widehat{X} = \overline{X} \setminus V(T^{\sigma}; \sigma \in \Sigma), \quad T^{\sigma} = T_1^{\varepsilon_1} \cdots T_r^{\varepsilon_r}, \quad \varepsilon_i = \begin{cases} 1, & v_i \notin \sigma, \\ 0, & v_i \in \sigma. \end{cases}$$

Now, consider the dual map  $P^*: M \rightarrow E$ , where  $M := \text{Hom}(N, \mathbb{Z})$ , set  $K := E/P^*(M)$  and denote by  $Q: E \rightarrow K$  the projection. We obtain a  $K$ -grading of the polynomial ring  $\mathbb{K}[T_1, \dots, T_r]$  by setting

$$\deg(T_i) := Q(e_i) \in K.$$

This gives an action of  $H := \text{Spec } \mathbb{K}[K]$  on  $\overline{X}$ . The set  $\widehat{X} \subseteq \overline{X}$  is invariant and  $p: \widehat{X} \rightarrow X$  is a good quotient. Moreover,

$$W := \overline{X} \setminus \bigcup_{i \neq j} V(T_i, T_j) \subseteq \widehat{X}$$

satisfies the conditions of Definition 1.37 for this action. Thus,  $p: \widehat{X} \rightarrow X$  is a characteristic space for  $X$ . In particular, we have

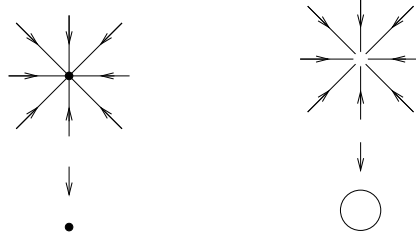
$$\text{Cl}(X) \cong K, \quad \mathcal{R}(X) \cong \mathbb{K}[T_1, \dots, T_r].$$

## 2. SECOND LECTURE

**2.1. Variation of good quotients.** Given a variety  $X$  with an action of a linearly reductive group  $G$ , the task of Geometric Invariant Theory is to describe the *good  $G$ -sets*, i.e. the invariant open subsets  $U \subseteq X$  admitting a good quotient  $U \rightarrow U//G$ . In general, there will be several such good  $G$ -sets; this effect is also called “variation of good quotients”. For details, see [4, Sec. III.1] and the original references [31, 9, 5].

**Example 2.1.** For the  $\mathbb{K}^*$ -action  $t \cdot (z_1, z_2) = (t^a z_1, t^b z_2)$  on  $\mathbb{K}^2$ ; as in 1.27, we consider the three typical cases.

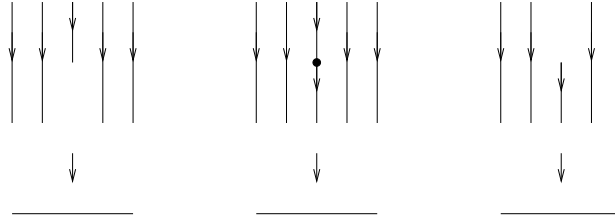
- (i) Let  $a = b = 1$ . Besides the good quotient  $\mathbb{K}^2 \rightarrow \{\text{pt}\}$ , there is a nice geometric quotient  $\mathbb{K}^2 \setminus \{0\} \rightarrow \mathbb{P}_1$ ,  $(z_1, z_2) \mapsto [z_1, z_2]$ .



- (ii) Let  $a = 0$  and  $b = 1$ . Besides the good quotient  $\mathbb{K}^2 \rightarrow \mathbb{K}$ ,  $(z_1, z_2) \mapsto z_1$ , there is a geometric quotient  $\mathbb{K}^2 \setminus V(T_2) \rightarrow \mathbb{K}$ ,  $(z_1, z_2) \mapsto z_1$ .



- (iii) Let  $a = 1$  and  $b = -1$ . Besides the good quotient  $\mathbb{K}^2 \rightarrow \mathbb{K}$ ,  $(z_1, z_2) \mapsto z_1 z_2$ , there are two geometric ones:  $\mathbb{K}^2 \setminus V(T_i) \rightarrow \mathbb{K}$ ,  $(z_1, z_2) \mapsto z_1 z_2$ .



All the good  $G$ -sets occurring in this example are maximal w.r.t. inclusions of the following type: a subset  $U' \subseteq U$  of a good  $G$ -set  $U \subseteq X$  is called  *$G$ -saturated* in  $U$  if it satisfies  $U' = \pi^{-1}(\pi(U'))$ , where  $\pi: U \rightarrow U//G$  is the good quotient. Any good  $G$ -set is  $G$ -saturated in a maximal one and for a description it is reasonable to focus on the latter ones.

Our aim is to present concrete combinatorial descriptions for quasiprojective and for torically embeddable quotients of quasitorus actions on certain affine varieties. Let us fix the setting. By  $K$  we denote a finitely generated abelian group and we consider an affine  $K$ -graded  $\mathbb{K}$ -algebra

$$A = \bigoplus_{w \in K} A_w.$$

Then the quasitorus  $H := \text{Spec } \mathbb{K}[K]$  acts on the affine variety  $X := \text{Spec } A$ . Let  $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$  denote the rational vector space associated to  $K$ . Given  $w \in K$ , we write again  $w$  for the element  $w \otimes 1 \in K_{\mathbb{Q}}$ .

**Definition 2.2.** The *weight cone* of the  $K$ -graded algebra  $A$  is the convex polyhedral cone

$$\omega_X := \omega(A) = \text{cone}(w \in K; A_w \neq \{0\}) \subseteq K_{\mathbb{Q}}.$$

To every point  $x \in X$ , we associate its *orbit cone*, this is the convex polyhedral cone

$$\omega_x := \text{cone}(w \in K; f(x) \neq 0 \text{ for some } f \in A_w) \subseteq \omega_X.$$

In order to see that these cones are indeed polyhedral, let  $f_1, \dots, f_r$  be homogeneous generators for  $A$  and set  $w_i := \deg(f_i)$ . Then the weight cone  $\omega_X$  is generated by  $w_1, \dots, w_r$  and the orbit cone  $\omega_x$  is generated by those  $w_i$  with  $f_i(x) \neq 0$ . In particular, we see that  $\omega_X$  is the general orbit cone and that there are only finitely many orbit cones.

**Example 2.3.** We determine the orbit cones of  $\mathbb{K}^*$ -action  $t \cdot (z_1, z_2) = (t^a z_1, t^b z_2)$  on  $\mathbb{K}^2$ ; again, we consider the three typical cases:

- (i) We have  $a = b = 1$ . The weight cone is  $\omega_{\mathbb{K}^2} = \mathbb{Q}_{\geq 0}$  and the possible orbit cones are

$$\omega_{(0,0)} = \{0\}, \quad \omega_{(1,1)} = \mathbb{Q}_{\geq 0}.$$

- (ii) We have  $a = 0$  and  $b = 1$ . The weight cone is  $\omega_{\mathbb{K}^2} = \mathbb{Q}_{\geq 0}$  and the possible orbit cones are

$$\omega_{(0,0)} = \{0\}, \quad \omega_{(1,0)} = \mathbb{Q}_{\geq 0}.$$

- (iii) We have  $a = 1$  and  $b = -1$ . The weight cone is  $\omega_{\mathbb{K}^2} = \mathbb{Q}$  and the possible orbit cones are

$$\omega_{(0,0)} = \{0\}, \quad \omega_{(1,0)} = \mathbb{Q}_{\geq 0}, \quad \omega_{(0,1)} = \mathbb{Q}_{\leq 0}, \quad \omega_{(1,1)} = \mathbb{Q}.$$

In this example we considered a subtorus action on a toric variety, and we used the fact that it suffices to determine one orbit cone for each toric orbit. This idea can be generalized using equivariant embeddings and gives the following concrete recipes for computing orbit cones.

**Remark 2.4.** Fix a system of homogeneous generators  $\mathfrak{F} = (f_1, \dots, f_r)$  for our  $K$ -graded algebra  $A$ . Then  $H$  acts diagonally on  $\mathbb{K}^r$  via the characters  $\chi^{w_1}, \dots, \chi^{w_r}$ , where  $w_i = \deg(f_i)$ , and we have an  $H$ -equivariant closed embedding

$$X \rightarrow \mathbb{K}^r, \quad x \mapsto (f_1(x), \dots, f_r(x)).$$

With  $E = \mathbb{Z}^r$  and  $\gamma = \text{cone}(e_1, \dots, e_r)$ , we may identify  $\mathbb{K}[T_1, \dots, T_r]$  with  $\mathbb{K}[E \cap \gamma]$  and thus regard  $\mathbb{K}^r$  as the affine toric variety associated to the dual cone  $\delta := \gamma^\vee$ . For any  $\gamma_0 \preceq \gamma$  and  $\delta_0 := \gamma_0^\perp \cap \delta$ , the following statements are equivalent:

- (i) the product over all  $f_i$  with  $e_i \in \gamma_0$  lies not in  $\sqrt{\langle f_j; e_j \notin \gamma_0 \rangle} \subseteq A$ ,
- (ii) there is a point  $z \in X$  with  $z_i \neq 0 \Leftrightarrow e_i \in \gamma_0$  for all  $1 \leq i \leq r$ ,
- (iii) the toric orbit  $\mathbb{T}^r \cdot z_{\delta_0} \subseteq \mathbb{K}^r$  corresponding to  $\delta_0 \preceq \delta$  meets  $X$ ,
- (iv) the intersection  $\delta_0^\circ \cap \text{Trop}(X)$  with the tropical variety is non-empty.

In order to determine the orbit cones, denote by  $Q: E \rightarrow K$  the homomorphism sending  $e_i$  to  $w_i$ . Moreover, we call  $\mathfrak{F}$ -*faces* the  $\gamma_0 \preceq \gamma$  satisfying (i). Then the orbit cones of  $X$  are precisely the images  $Q(\gamma_0)$ , where  $\gamma_0 \preceq \gamma$  is an  $\mathfrak{F}$ -face.

**Example 2.5.** Set  $K := \mathbb{Z}^2$  and consider the  $K$ -grading of  $\mathbb{K}[T_1, \dots, T_5]$  defined by  $\deg(T_i) := w_i$ , where  $w_i$  is the  $i$ -th column of the matrix

$$Q := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}.$$

The corresponding action of  $H = \mathbb{T}^2$  on  $\mathbb{K}^5$  leaves  $X := V(T_1 T_2 + T_3^2 + T_4 T_5)$  invariant. The possible orbit cones  $\omega_x$  for  $x \in X$  are

$$\begin{aligned} &\{0\}, \quad \text{cone}(w_1), \quad \text{cone}(w_2), \quad \text{cone}(w_4), \quad \text{cone}(w_5), \\ &\text{cone}(w_1, w_4), \quad \text{cone}(w_2, w_4), \quad \text{cone}(w_1, w_5), \quad \text{cone}(w_2, w_5). \end{aligned}$$

**Definition 2.6.** The *GIT-cone* of an element  $w \in \omega_X$  is the (nonempty) intersection of all orbit cones containing it:

$$\lambda(w) := \bigcap_{\substack{x \in X, \\ w \in \omega_x}} \omega_x.$$

We write  $\Lambda(X, H)$  for the set of all GIT-cones. The *set of semistable points* associated to a GIT-cone  $\lambda \subseteq \omega_X$  is

$$X^{ss}(\lambda) = \{x \in X; \lambda \subseteq \omega_x\} \subseteq X.$$

**Remark 2.7.** Given a GIT-cone  $\lambda \in \Lambda(X, H)$  and any weight  $w \in \lambda^\circ$  in its relative interior, one easily checks

$$X^{ss}(\lambda) = \{x \in X; f(x) \neq 0 \text{ for some } f \in A_{nw}, n > 0\}.$$

That means that  $X^{ss}(\lambda)$  is the set of semistable points associated to the linearization of the trivial bundle given by the character  $\chi^w$  in the sense of Mumford.

**Example 2.8.** We compute GIT-cones and associated sets of semistable points for the  $\mathbb{K}^*$ -action  $t \cdot (z, w) = (t^a z, t^b w)$  on  $\mathbb{K}^2$  in the three typical cases.

$a = 1, b = 1 :$	$\lambda(0) = \{0\}$	$X^{ss}(\lambda(0)) = \mathbb{K}^2,$
	$\lambda(1) = \mathbb{Q}_{\geq 0}$	$X^{ss}(\lambda(1)) = \mathbb{K}^2 \setminus \{0\},$
$a = 0, b = 1 :$	$\lambda(0) = \{0\}$	$X^{ss}(\lambda(0)) = \mathbb{K}^2,$
	$\lambda(1) = \mathbb{Q}_{\geq 0}$	$X^{ss}(\lambda(1)) = \mathbb{K}^2 \setminus V(T_2),$
$a = -1, b = 1 :$	$\lambda(0) = \{0\}$	$X^{ss}(\lambda(0)) = \mathbb{K}^2,$
	$\lambda(-1) = \mathbb{Q}_{\leq 0}$	$X^{ss}(\lambda(-1)) = \mathbb{K}^2 \setminus V(T_1),$
	$\lambda(1) = \mathbb{Q}_{\geq 0}$	$X^{ss}(\lambda(1)) = \mathbb{K}^2 \setminus V(T_2).$

In the following statement, we mean by a *quasifan* a finite collection  $\Lambda$  of not necessarily pointed polyhedral cones in a rational vector space such that any two  $\lambda_1, \lambda_2 \in \Lambda$  intersect in a common face and for  $\lambda \in \Lambda$  also every face of  $\lambda$  belongs to  $\Lambda$ .

**Theorem 2.9.** *The collection  $\Lambda(X, H) = \{\lambda(w); w \in \omega_X\}$  of all GIT-cones is a quasifan in  $K_{\mathbb{Q}}$  having the weight cone  $\omega_X$  as its support.*

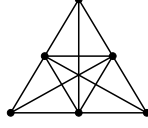
- (i) *For every  $\lambda \in \Lambda(X, H)$ , there is a good quotient  $X^{ss}(\lambda) \rightarrow Y(\lambda)$  for the action of  $H$  on  $X^{ss}(\lambda)$ .*
- (ii) *For any two GIT-cones  $\lambda_1, \lambda_2 \in \Lambda(X, H)$ , we have  $\lambda_2 \preceq \lambda_1$  if and only if  $X^{ss}(\lambda_1) \subseteq X^{ss}(\lambda_2)$  holds.*
- (iii) *If  $X^{ss}(\lambda_1) \subseteq X^{ss}(\lambda_2)$  holds, then there is an induced projective morphism  $Y(\lambda_1) \rightarrow Y(\lambda_2)$ ; in particular, every  $Y(\lambda)$  is projective over  $Y(0)$ .*

The collection  $\Lambda(X, H)$  is called the *GIT-(quasi-)fan* of the  $H$ -variety  $X$ . For a diagonal  $H$ -action on  $\mathbb{K}^r$ , the orbit cones are  $\text{cone}(\deg(T_i); i \in I)$ , where  $I$  runs through the subsets of  $\{1, \dots, r\}$  and, thus the GIT-fan equals the Gelfand-Kapranov-Zelevinsky decomposition associated to  $\deg(T_1), \dots, \deg(T_r) \in K_{\mathbb{Q}}$ .

**Example 2.10.** Consider the action of the standard three torus  $\mathbb{T}^3$  on  $\mathbb{K}^6$  defined by  $\deg(T_i) = w_i$ , where  $w_1, \dots, w_6$  are defined as

$$\begin{aligned} w_1 &= (1, 0, 0), & w_2 &= (0, 1, 0), & w_3 &= (0, 0, 1), \\ w_4 &= (1, 1, 0), & w_5 &= (1, 0, 1), & w_6 &= (0, 1, 1). \end{aligned}$$

Then the GIT-fan subdivides the positive orthant in  $\mathbb{Q}^3$ ; intersecting with a suitable plane perpendicular to the line through  $(1, 1, 1)$  gives the following picture.



If every  $H$ -invariant divisor on  $X$  is principal, then the GIT-fan controls the whole variation of good quotients with a quasiprojective quotient space. For the precise statement let us call a good  $H$ -set  $U \subseteq X$  *qp-maximal* if  $U//H$  is quasiprojective and  $U$  is maximal w.r.t.  $H$ -saturated inclusion among all good  $H$ -sets  $W \subseteq X$  with  $W//H$  quasiprojective.

**Theorem 2.11.** *Assume that  $X$  is normal and for every  $H$ -invariant divisor on  $X$  some positive multiple is principal. Then, with the GIT-fan  $\Lambda(X, H)$  of the  $H$ -action on  $X$ , we have mutually inverse order reversing bijections*

$$\begin{aligned} \Lambda(X, H) &\longleftrightarrow \{ \text{qp-maximal subsets of } X \} \\ \lambda &\mapsto X^{ss}(\lambda) = \{ x \in X; \lambda \subseteq \omega_x \} \\ \bigcap_{x \in U} \omega_x =: \lambda(U) &\longleftrightarrow U. \end{aligned}$$

Now we look for more general quotient spaces. We say that a variety  $X$  has the  $A_2$ -property, if any two points  $x, x' \in X$  admit a common affine open neighborhood in  $X$ . By [44], the normal  $A_2$ -varieties are precisely those that admit a closed embedding into a toric variety.

**Definition 2.12.** Let  $\Omega_X$  denote the collection of all orbit cones  $\omega_x$ , where  $x \in X$ . A *bunch of orbit cones* is a nonempty collection  $\Phi \subseteq \Omega_X$  such that

- (i) given  $\omega_1, \omega_2 \in \Phi$ , one has  $\omega_1^\circ \cap \omega_2^\circ \neq \emptyset$ ,
- (ii) given  $\omega \in \Phi$ , every orbit cone  $\omega_0 \in \Omega_X$  with  $\omega^\circ \subseteq \omega_0^\circ$  belongs to  $\Phi$ .

A *maximal bunch of orbit cones* is a bunch of orbit cones  $\Phi \subseteq \Omega_X$  which cannot be enlarged by adding further orbit cones.

**Definition 2.13.** Let  $\Phi, \Phi' \subseteq \Omega_X$  be bunches of orbit cones. We say that  $\Phi$  *refines*  $\Phi'$  (written  $\Phi \leq \Phi'$ ), if for any  $\omega' \in \Phi'$  there is an  $\omega \in \Phi$  with  $\omega \subseteq \omega'$ .

**Example 2.14.** Consider once more the action of the standard three torus  $\mathbb{T}^3$  on  $\mathbb{K}^6$  discussed in 2.10. Here are two maximal bunches, indicated by drawing their minimal members:



**Definition 2.15.** To any collection of orbit cones  $\Phi$  of  $X$ , we associate the following subset of  $X$ :

$$U(\Phi) := \{ x \in X; \omega_0 \preceq \omega_x \text{ for some } \omega_0 \in \Phi \}.$$

Conversely, to any  $H$ -invariant subset  $U \subseteq X$ , we associate the following collection of orbit cones

$$\Phi(U) := \{ \omega_x; x \in U \text{ with } H \cdot x \text{ closed in } U \}.$$

By an  $(H, 2)$ -maximal subset of  $X$  we mean a good  $H$ -set  $U \subseteq X$  with  $U//H$  an  $A_2$ -variety such that  $U$  is maximal w.r.t.  $H$ -saturated inclusion among all good  $H$ -sets  $W \subseteq X$  with  $W//H$  an  $A_2$ -variety. We are ready for the result.

**Theorem 2.16.** *Assume that  $X$  is normal and for every  $H$ -invariant divisor on  $X$  some positive multiple is principal. Then we have mutually inverse order reversing bijections*

$$\begin{aligned} \{\text{maximal bunches of orbit cones in } \Omega_X\} &\longleftrightarrow \{(H, 2)\text{-maximal subsets of } X\} \\ \Phi &\mapsto U(\Phi) \\ \Phi(U) &\leftarrow U. \end{aligned}$$

**Remark 2.17.** Every GIT-chamber  $\lambda \in \Lambda(X, H)$  defines a bunch of orbit cones  $\Phi(\lambda) = \{\omega_x; \lambda^\circ \subseteq \omega^\circ\}$ . These bunches turn out to be maximal and they correspond to the qp-maximal subsets of  $X$ ; in particular, the latter ones are  $(H, 2)$ -maximal. The bunches of 2.14 give rise to non-projective complete quotients.

**2.2. Cox rings and combinatorics.** Here we present the combinatorial approach to varieties with finitely generated Cox ring developed in [10, 19], see also [4, Chap. III]. The approach generalizes the combinatorial description of toric varieties and has many common features with methods of [17, 26] for investigating subvarieties of weighted complete spaces. The whole thing is based on the following simple observation.

**Remark 2.18.** Let  $X$  be a normal variety with  $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$  and finitely generated divisor class group. If the Cox ring  $\mathcal{R}(X)$  is finitely generated, then we obtain the following picture

$$\begin{array}{ccccc} \mathrm{Spec}_X \mathcal{R} & = & \widehat{X} & \subseteq & \overline{X} & = & \mathrm{Spec} \mathcal{R}(X) \\ & & \downarrow \scriptstyle // H_X & & & & \\ & & X & & & & \end{array}$$

where  $\widehat{X} \subseteq \overline{X}$  is an open  $H_X$ -invariant subset of the  $H_X$ -factorial affine variety  $\overline{X}$  and the characteristic space  $\widehat{X} \rightarrow X$  is a good quotient for the  $H_X$ -action. We call the affine  $H_X$ -variety  $\overline{X}$  the *total coordinate space* of  $X$ .

Thus, we see that all varieties sharing the same divisor class group  $K$  and finitely generated Cox ring  $R$  occur as good quotients of suitable open subsets of  $\mathrm{Spec} R$  by the action of  $\mathrm{Spec} \mathbb{K}[K]$ . The latter ones we just described in combinatorial terms via Geometric Invariant Theory. We now turn this picture into a combinatorial language allowing explicit computations.

**Definition 2.19.** Let  $K$  be a finitely generated abelian group and  $R$  a factorially  $K$ -graded affine algebra with  $R^* = \mathbb{K}^*$ . Moreover, let  $\mathfrak{F} = (f_1, \dots, f_r)$  be a system of pairwise nonassociated  $K$ -prime generators for  $R$ .

- (i) The *projected cone* associated to  $\mathfrak{F}$  is  $(E \xrightarrow{Q} K, \gamma)$ , where  $E := \mathbb{Z}^r$ , the homomorphism  $Q: E \rightarrow K$  sends the  $i$ -th canonical basis vector  $e_i \in E$  to  $w_i := \deg(f_i) \in K$  and  $\gamma \subseteq E_{\mathbb{Q}}$  is the convex cone generated by  $e_1, \dots, e_r$ .
- (ii) We say that the  $K$ -grading of  $R$  is *almost free* if for every facet  $\gamma_0 \preceq \gamma$  the image  $Q(\gamma_0 \cap E)$  generates the abelian group  $K$ .
- (iii) We say that  $\gamma_0 \preceq \gamma$  is an  *$\mathfrak{F}$ -face*, if the product over all  $f_i$  with  $e_i \in \gamma_0$  does not lie in  $\sqrt{\langle f_j; e_j \notin \gamma_0 \rangle} \subseteq A$ .
- (iv) Let  $\Omega_{\mathfrak{F}} = \{Q(\gamma_0); \gamma_0 \preceq \gamma \text{ } \mathfrak{F}\text{-face}\}$  denote the collection of projected  $\mathfrak{F}$ -faces. An  *$\mathfrak{F}$ -bunch* is a nonempty subset  $\Phi \subseteq \Omega_{\mathfrak{F}}$  such that
  - (a) for any two  $\tau_1, \tau_2 \in \Phi$ , we have  $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$ ,
  - (b) if  $\tau_1^\circ \subseteq \tau^\circ$  holds for  $\tau_1 \in \Phi$  and  $\tau \in \Omega_{\mathfrak{F}}$ , then  $\tau \in \Phi$  holds.
- (v) We say that an  $\mathfrak{F}$ -bunch  $\Phi$  is *true* if for every facet  $\gamma_0 \preceq \gamma$  the image  $Q(\gamma_0)$  belongs to  $\Phi$ .



**Definition 2.20.** A *bunched ring* is a triple  $(R, \mathfrak{F}, \Phi)$ , where  $R$  is an almost freely factorially  $K$ -graded affine  $\mathbb{K}$ -algebra such that  $R^* = \mathbb{K}^*$  holds,  $\mathfrak{F}$  is a system of pairwise non-associated  $K$ -prime generators for  $R$  and  $\Phi$  is a true  $\mathfrak{F}$ -bunch.

**Construction 2.21.** Let  $(R, \mathfrak{F}, \Phi)$  be a bunched ring. Then  $\Phi$  is a bunch of orbit cones for the action of  $H := \operatorname{Spec} \mathbb{K}[K]$  on  $\overline{X} := \operatorname{Spec} R$ . Thus, we have the associated open set and its quotient

$$\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi) = \overline{X}(\Phi) \subseteq \overline{X},$$

$$X := X(R, \mathfrak{F}, \Phi) := \widehat{X}(R, \mathfrak{F}, \Phi) // H.$$

We denote the quotient map by  $p: \widehat{X} \rightarrow X$ . Conditions 2.19 (ii) and (v) ensure that the  $H$ -action on  $\widehat{X}$  is strongly stable. Moreover, every member  $f_i$  of  $\mathfrak{F}$  defines a prime divisor  $D_X^i := p(V(\widehat{X}, f_i))$  on  $X$ .

**Theorem 2.22.** Let  $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$  and  $X := X(R, \mathfrak{F}, \Phi)$  arise from a bunched ring  $(R, \mathfrak{F}, \Phi)$ . Then  $X$  is a normal  $A_2$ -variety with

$$\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}}), \quad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*,$$

there is an isomorphism  $\operatorname{Cl}(X) \rightarrow K$  sending  $[D_X^i]$  to  $\deg(f_i)$ , the map  $p: \widehat{X} \rightarrow X$  is a characteristic space and the Cox ring  $\mathcal{R}(X)$  is isomorphic to  $R$ .

**Theorem 2.23.** Every complete normal  $A_2$ -variety with finitely generated Cox ring arises from a bunched ring; in particular, every projective normal variety with finitely generated Cox ring does so.

Let us illustrate Construction 2.21 with two examples. The first one shows how toric varieties fit into the picture of bunched rings.

**Example 2.24** (Bunched polynomial rings). Consider a bunched ring  $(R, \mathfrak{F}, \Phi)$  with  $R = \mathbb{K}[T_1, \dots, T_r]$  and  $\mathfrak{F} := (T_1, \dots, T_r)$ . Then  $X(R, \mathfrak{F}, \Phi)$  is a toric variety. Its defining fan  $\Sigma$  is obtained from  $\Phi$  via linear Gale duality:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_{\mathbb{Q}} & \xrightarrow{Q^*} & F_{\mathbb{Q}} & \xrightarrow{P} & N_{\mathbb{Q}} \longrightarrow 0 \\ & & & & \uparrow \Sigma^\dagger & \xrightarrow{\delta_0 \mapsto P(\delta_0)} & \Sigma \\ & & & & \uparrow \gamma_0 \mapsto \gamma_0^\perp \cap \delta & & \\ & & \Phi & \xleftarrow{Q(\gamma_0) \leftarrow \gamma_0} & \Phi^\dagger & & \\ 0 & \longleftarrow & K_{\mathbb{Q}} & \xleftarrow{Q} & E_{\mathbb{Q}} & \xleftarrow{P^*} & M_{\mathbb{Q}} \longleftarrow 0 \end{array}$$

Here  $\Phi^\dagger$  consists of those faces of the orthant  $\gamma \subseteq E_{\mathbb{Q}}$  that map onto a member of  $\Phi$  and  $\Sigma^\dagger$  of the corresponding faces of the dual orthant  $\delta \subseteq F_{\mathbb{Q}}$ . Note that  $P: F \rightarrow N$  is the same map as in 1.43 and the  $D_X^i$  are exactly the toric prime divisors.

**Example 2.25** (A singular del Pezzo surface). Consider  $K := \mathbb{Z}^2$  and the  $K$ -grading of  $\mathbb{K}[T_1, \dots, T_5]$  given by  $\deg(T_i) := w_i$ , where  $w_i$  is the  $i$ -th column of

$$Q := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

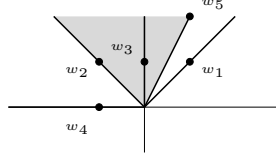
Then this  $K$ -grading descends to a  $K$ -grading of the following residue algebra which is known to be factorial:

$$R := \mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3^2 + T_4 T_5 \rangle.$$

The classes  $f_i \in R$  of  $T_i \in \mathbb{K}[T_1, \dots, T_5]$ , where  $1 \leq i \leq 5$ , form a system  $\mathfrak{F}$  of pairwise nonassociated  $K$ -prime generators of  $R$ . We have

$$E = \mathbb{Z}^5, \quad \gamma = \operatorname{cone}(e_1, \dots, e_5)$$

and the  $K$ -grading is almost free. Computing the  $\mathfrak{F}$ -faces, we see that there is one maximal true  $\mathfrak{F}$ -bunch  $\Phi$ ; it has  $\tau := \text{cone}(w_2, w_5)$  as its unique minimal cone.



Note that  $\tau$  is a GIT-cone,  $\Phi = \Phi(\tau)$  holds with  $\Phi(\tau)$  as in 2.17 and  $\hat{X}(R, \mathfrak{F}, \Phi)$  equals  $\overline{X}^{ss}(\tau)$  in  $\overline{X} = V(\mathbb{K}^5; T_1T_2 + T_3^2 + T_4T_5)$ . For  $X = X(R, \mathfrak{F}, \Phi)$  we have

$$\dim(X) = 2, \quad \text{Cl}(X) = \mathbb{Z}^2, \quad \mathcal{R}(X) = R.$$

**Definition 2.26.** Let  $(R, \mathfrak{F}, \Phi)$  be a bunched ring and  $(E \xrightarrow{Q} K, \gamma)$  its projected cone. The *collection of relevant faces* and the *covering collection* are

$$\begin{aligned} \text{rlv}(\Phi) &:= \{\gamma_0 \preceq \gamma; \gamma_0 \text{ an } \mathfrak{F}\text{-face with } Q(\gamma_0) \in \Phi\}, \\ \text{cov}(\Phi) &:= \{\gamma_0 \in \text{rlv}(\Phi); \gamma_0 \text{ minimal}\}. \end{aligned}$$

**Construction 2.27** (Canonical toric embedding). Any bunched ring  $(R, \mathfrak{F}, \Phi)$  defines a bunched polynomial ring  $(R', \mathfrak{F}', \Phi')$  by “forgetting the relations”: If the system of generators of  $R$  is  $\mathfrak{F} = (f_1, \dots, f_r)$ , set

$$R' := \mathbb{K}[T_1, \dots, T_r], \quad \deg(T_i) := \deg(f_i) \in K, \quad \mathfrak{F}' := (T_1, \dots, T_r)$$

and let  $\Phi'$  be the  $\mathfrak{F}'$ -bunch generated by  $\Phi$ , i.e. it consists of all projected faces  $Q(\gamma_0)$  with  $\tau^\circ \subseteq Q(\gamma_0)^\circ$  for some  $\tau \in \Phi$ . Then we obtain a commutative diagram, where the induced map of quotients  $\iota: X \rightarrow Z$  is a closed embedding of the varieties  $X$  and  $Z$  associated to the bunched rings  $(R, \mathfrak{F}, \Phi)$  and  $(R', \mathfrak{F}', \Phi')$  respectively:

$$\begin{array}{ccccc} \overline{X} & \supseteq & \hat{X} & \longrightarrow & \hat{Z} & \subseteq & \overline{Z} \\ & & \parallel H \downarrow & & \downarrow \parallel H & & \\ & & X & \xrightarrow{\iota} & Z & & \end{array}$$

By construction,  $Z$  is toric and we have an isomorphism  $\iota^*: \text{Cl}(Z) \rightarrow \text{Cl}(X)$ . Then, in the setting of 2.24, the toric orbits of  $Z$  intersecting  $X$  are precisely the orbits  $B(\sigma) \subseteq Z$  corresponding to cones  $\sigma = P(\gamma_0^*)$  with  $\gamma_0 \in \text{rlv}(\Phi)$ . In particular, we obtain a decomposition into locally closed strata

$$X = \bigcup_{\gamma_0 \in \text{rlv}(\Phi)} X(\gamma_0), \quad X(\gamma_0) := X \cap B(\sigma).$$

**Remark 2.28.** In general, the canonical toric ambient variety  $Z$  is not complete, even if  $X$  is. If  $X$  is projective, then  $\Phi = \Phi(\lambda)$  holds with a GIT-cone  $\lambda \in \Lambda(\overline{X}, H)$ . The toric GIT-fan  $\Lambda(\overline{Z}, H)$  refines  $\Lambda(\overline{X}, H)$  and every  $\eta \in \Lambda(\overline{Z}, H)$  with  $\eta^\circ \subseteq \lambda^\circ$  defines a projective completion of  $Z$ . For example, in the setting of 2.25, the two GIT-fans are



The GIT-cones  $\text{cone}(w_2, w_3)$  and  $\text{cone}(w_3, w_5)$  in  $\Lambda(\overline{Z}, H)$  provide completions of  $Z$  by  $\mathbb{Q}$ -factorial projective toric varieties  $Z_1$  and  $Z_2$  and  $\text{cone}(w_3)$  gives a completion by a projective toric variety  $Z_3$  with a non- $\mathbb{Q}$ -factorial singularity.

We now indicate how to read off basic geometric properties from defining data. In the sequel,  $X$  is the variety arising from a bunched ring  $(R, \mathfrak{F}, \Phi)$ .

**Theorem 2.29.** *Consider a relevant face  $\gamma_0 \in \text{rlv}(\Phi)$  and a point  $x \in X(\gamma_0)$ . Then we have a commutative diagram*

$$\begin{array}{ccc} \text{Cl}(X) & \longrightarrow & \text{Cl}(X, x) \\ \cong \updownarrow & & \updownarrow \cong \\ K & \longrightarrow & K/Q(\text{lin}(\gamma_0) \cap E) \end{array}$$

*In particular, the local divisor class groups are constant along the pieces  $X(\gamma_0)$ , where  $\gamma_0 \in \text{rlv}(\Phi)$ . Moreover, the Picard group of  $X$  is given by*

$$\text{Pic}(X) \cong \bigcap_{\gamma_0 \in \text{cov}(\Phi)} Q(\text{lin}(\gamma_0) \cap E).$$

**Theorem 2.30.** *Consider a relevant face  $\gamma_0 \in \text{rlv}(\Phi)$  and point  $x \in X(\gamma_0)$  in the corresponding stratum.*

- (i) *The point  $x$  is factorial if and only if  $Q$  maps  $\text{lin}(\gamma_0) \cap E$  onto  $K$ .*
- (ii) *The point  $x$  is  $\mathbb{Q}$ -factorial if and only if  $Q(\gamma_0)$  is of full dimension.*

*In particular,  $X$  is  $\mathbb{Q}$ -factorial, if and only if  $\Phi$  consists of full-dimensional cones. If  $\hat{X}$  is smooth, then every factorial point of  $X$  is smooth.*

**Theorem 2.31.** *In the divisor class group  $K = \text{Cl}(X)$ , we have the following descriptions of the cones of effective, movable, semiample and ample divisors:*

$$\begin{aligned} \text{Eff}(X) &= Q(\gamma), & \text{Mov}(X) &= \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0), \\ \text{Sample}(X) &= \bigcap_{\tau \in \Phi} \tau, & \text{Ample}(X) &= \bigcap_{\tau \in \Phi} \tau^\circ. \end{aligned}$$

**Example 2.32** (The singular del Pezzo surface, continued). The variety  $X$  of 2.25 is a  $\mathbb{Q}$ -factorial surface. It has a single singularity, namely the point in the piece  $x_0 \in X(\gamma_0)$  for  $\gamma_0 = \text{cone}(e_2, e_5)$ . The local class group  $\text{Cl}(X, x_0)$  is cyclic of order three and the Picard group of  $X$  is of index 3 in  $\text{Cl}(X)$ . Moreover, the ample cone of  $X$  is generated by  $w_2$  and  $w_5$ . In particular,  $X$  is projective.

**Remark 2.33.** Applying 2.29, 2.30 and 2.31 to the canonical toric ambient variety  $Z$  shows that  $X$  inherits local class groups and singularities from  $Z$  and the Picard group as well as the various cones of divisors of  $X$  and  $Z$  coincide. However, factorial singularities of  $X$  are smooth points of  $Z$ , see 3.21 for an example. Moreover,  $\text{Pic}(X) = \text{Pic}(Z)$  and  $\text{Ample}(X) = \text{Ample}(Z)$  can get lost when replacing  $Z$  with a completion.

The following two statements concern the case that  $R$  is a complete intersection in the sense that with  $d := r - \dim(X) - \dim(K)$ , there are  $K$ -homogeneous generators  $g_1, \dots, g_d$  for the ideal of relations between  $f_1, \dots, f_r$ . Set  $w_i := \deg(f_i)$  and  $u_j := \deg(g_j)$ .

**Theorem 2.34.** *Suppose that  $R$  is a complete intersection as above. Then the canonical class of  $X$  is given in  $K = \text{Cl}(X)$  by*

$$\mathcal{K}_X = \sum u_j - \sum w_i.$$

**Remark 2.35** (Computing intersection numbers). Suppose that  $R$  is a complete intersection and that  $\Phi = \Phi(\lambda)$  holds with a full-dimensional  $\lambda \in \Lambda(\overline{X}, H)$ . Fix a full-dimensional  $\eta \in \Lambda(\overline{Z}, H)$  with  $\eta^\circ \subseteq \lambda^\circ$ . For  $w_{i_1}, \dots, w_{i_{n+d}}$  let  $w_{j_1}, \dots, w_{j_{r-n-d}}$  denote the complementary weights and set

$$\begin{aligned}\tau(w_{i_1}, \dots, w_{i_{n+d}}) &:= \text{cone}(w_{j_1}, \dots, w_{j_{r-n-d}}), \\ \mu(w_{i_1}, \dots, w_{i_{n+d}}) &:= [K : \langle w_{j_1}, \dots, w_{j_{r-n-d}} \rangle].\end{aligned}$$

Then the intersection product  $K_{\mathbb{Q}}^{n+d} \rightarrow \mathbb{Q}$  of the ( $\mathbb{Q}$ -factorial) toric variety  $Z_1$  associated to  $\Phi(\eta)$  is determined by the values

$$w_{i_1} \cdots w_{i_{n+d}} = \begin{cases} \mu(w_{i_1}, \dots, w_{i_{n+d}})^{-1}, & \eta \subseteq \tau(w_{i_1}, \dots, w_{i_{n+d}}), \\ 0, & \eta \not\subseteq \tau(w_{i_1}, \dots, w_{i_{n+d}}). \end{cases}$$

As a complete intersection,  $X \subseteq Z_1$  inherits intersection theory. For a tuple  $D_X^{i_1}, \dots, D_X^{i_n}$  on  $X$ , its intersection number can be computed by

$$D_X^{i_1} \cdots D_X^{i_n} = w_{i_1} \cdots w_{i_n} \cdot u_1 \cdots u_d.$$

**Example 2.36** (The singular del Pezzo surface, continued). Consider once more the surface  $X$  of 2.25. The degree of the defining relation is  $\deg(T_1 T_2 + T_3^2 + T_4 T_5) = 2w_3$  and thus the canonical class of  $X$  is given as

$$\mathcal{K}_X = 2w_3 - (w_1 + w_2 + w_3 + w_4 + w_5) = -3w_3.$$

In particular, we see that the anticanonical class is ample and thus  $X$  is a (singular) del Pezzo surface. The self intersection number of the canonical class is

$$\mathcal{K}_X^2 = (3w_3)^2 = \frac{9(w_1 + w_2)(w_4 + w_5)}{4}.$$

The  $w_i \cdot w_j$  equal the toric intersection numbers  $2w_i \cdot w_j \cdot w_3$ . In order to compute them, let  $w_{ij}^1, w_{ij}^2$  denote the weights in  $\{w_1, \dots, w_5\} \setminus \{w_i, w_j, w_3\}$ . Then we have

$$w_i \cdot w_j \cdot w_3 = \begin{cases} \mu(w_i, w_j, w_3)^{-1}, & \tau \subseteq \text{cone}(w_{ij}^1, w_{ij}^2), \\ 0, & \tau \not\subseteq \text{cone}(w_{ij}^1, w_{ij}^2), \end{cases}$$

where the multiplicity  $\mu(w_i, w_j, w_3)$  is the absolute value of  $\det(w_{ij}^1, w_{ij}^2)$ . Thus, we can proceed in the computation:

$$\mathcal{K}_X^2 = (3w_3)^2 = \frac{9 \cdot 2}{4} (|\det(w_2, w_5)|^{-1} + |\det(w_1, w_4)|^{-1}) = \frac{9}{2} \cdot \frac{4}{3} = 6.$$

**2.3. Mori dream spaces.** We take a closer look to the  $\mathbb{Q}$ -factorial projective varieties with a finitely generated Cox ring. Hu and Keel [25] called them *Mori dream spaces* and characterized them in terms of cones of divisors. In the context of normal complete varieties their statement reads as follows; by a *small birational map* we mean a rational map defining an isomorphism of open subsets with complement of codimension two.

**Theorem 2.37.** *Let  $X$  be a normal complete variety with finitely generated divisor class group. Then the following statements are equivalent.*

- (i) *The Cox ring  $\mathcal{R}(X)$  is finitely generated.*
- (ii) *There are small birational maps  $\pi_i: X \rightarrow X_i$ , where  $i = 1, \dots, r$ , such that each semiample cone  $\text{SAmple}(X_i) \subseteq \text{Cl}_{\mathbb{Q}}(X)$  is polyhedral and one has*

$$\text{Mov}(X) = \pi_1^*(\text{SAmple}(X_1)) \cup \dots \cup \pi_r^*(\text{SAmple}(X_r)).$$

*Moreover, if one of these two statements holds, then there is a small birational map  $X \rightarrow X'$  with a  $\mathbb{Q}$ -factorial projective variety  $X'$ .*

There are many examples of Mori dream spaces. Besides the toric and more generally spherical varieties, all unirational varieties with a reductive group action of complexity one are Mori dream spaces. Other important examples are the log terminal Fano varieties [11]. Moreover, K3- and Enriques surfaces are Mori dream spaces if and only if their effective cone is polyhedral [3, 43]; the same holds in higher dimensions for Calabi-Yau varieties [29]. Specializing to surfaces, the above theorem gives the following.

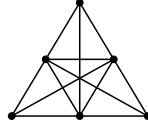
**Corollary 2.38.** *Let  $X$  be a normal complete surface with finitely generated divisor class group  $\text{Cl}(X)$ . Then the following statements are equivalent.*

- (i) *The Cox ring  $\mathcal{R}(X)$  is finitely generated.*
- (ii) *One has  $\text{Mov}(X) = \text{SAmple}(X)$  and this cone is polyhedral.*

*Moreover, if one of these two statements holds, then the surface  $X$  is  $\mathbb{Q}$ -factorial and projective.*

The Mori dream spaces sharing a given Cox ring fit into a nice picture in terms of the GIT-fan; by the *moving cone* of the  $K$ -graded algebra  $R$  we mean here the intersection  $\text{Mov}(R)$  over all  $\text{cone}(w_1, \dots, \widehat{w_i}, \dots, w_r)$ , where the  $w_i$  are the degrees of any system of pairwise nonassociated homogeneous  $K$ -prime generators for  $R$ .

**Remark 2.39.** Let  $R = \bigoplus_K R_w$  be an almost freely factorially graded affine algebra with  $R_0 = \mathbb{K}$  and consider the GIT-fan  $\Lambda(\overline{X}, H)$  of the action of  $H = \text{Spec } \mathbb{K}[K]$  on  $\overline{X} = \text{Spec } R$ .



Then every GIT-cone  $\lambda \in \Lambda(\overline{X}, H)$  defines a projective variety  $X(\lambda) = \overline{X}^{ss}(\lambda) // H$ . If  $\lambda^\circ \subseteq \text{Mov}(R)^\circ$  holds, then  $X(\lambda)$  is the variety associated to the bunched ring  $(R, \mathfrak{F}, \Phi(\lambda))$  with  $\Phi(\lambda)$  defined as in 2.17. In particular, in this case we have

$$\text{Cl}(X(\lambda)) = K, \quad \mathcal{R}(X(\lambda)) = R,$$

$$\text{Mov}(X(\lambda)) = \text{Mov}(R), \quad \text{SAmple}(X(\lambda)) = \lambda.$$

All projective varieties with Cox ring  $R$  are isomorphic to some  $X(\lambda)$  with  $\lambda^\circ \subseteq \text{Mov}(R)^\circ$  and the Mori dream spaces among them are precisely those arising from a full dimensional  $\lambda$ .

Let  $X$  be the variety arising from a bunched ring  $(R, \mathfrak{F}, \Phi)$ . Every Weil divisor  $D$  on  $X$  defines a positively graded sheaf

$$\mathcal{S}^+ := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{S}_n^+, \quad \mathcal{S}_n^+ := \mathcal{O}_X(nD).$$

The algebra of global sections of this sheaf inherits finite generation from the Cox ring. In particular, we obtain a rational map

$$\varphi(D): X \rightarrow X(D), \quad X(D) := \text{Proj}(\Gamma(X, \mathcal{S}^+)).$$

Note that  $X(D)$  is explicitly given as the closure of the image of the rational map  $X \rightarrow \mathbb{P}_m$  determined by the linear system of a sufficiently big multiple  $nD$ .

**Remark 2.40.** Consider the GIT-fan  $\Lambda(\overline{X}, H)$  of the action of  $H = \text{Spec } \mathbb{K}[K]$  on  $\overline{X} = \text{Spec } R$ . Let  $\lambda \in \Lambda(\overline{X}, H)$  be the cone with  $[D] \in \lambda^\circ$  and  $W \subseteq \overline{X}$  the open subset obtained by removing the zero sets of the generators  $f_1, \dots, f_r \in R$ . Then

we obtain a commutative diagram

$$\begin{array}{ccccc}
 \widehat{X} & \supseteq & W & \subseteq & \overline{X}^{ss}(\lambda) \\
 \parallel H \downarrow & & \downarrow & & \downarrow \parallel H \\
 X & \supseteq & W/H & \longrightarrow & X(\lambda) \\
 & \searrow & & & \parallel \\
 & & \varphi(D) & \dashrightarrow & X(D)
 \end{array}$$

**Proposition 2.41.** *Let  $D \in \text{WDiv}(X)$  be any Weil divisor, and denote by  $[D] \in \text{Cl}(X)$  its class. Then the associated rational map  $\varphi(D): X \rightarrow X(D)$  is*

- (i) *birational if and only if  $[D] \in \text{Eff}(X)^\circ$  holds,*
- (ii) *small birational if and only if  $[D] \in \text{Mov}(X)^\circ$  holds,*
- (iii) *a morphism if and only if  $[D] \in \text{SAmple}(X)$  holds,*
- (iv) *an isomorphism if and only if  $[D] \in \text{Ample}(X)$  holds.*

**Remark 2.42.** The observations made so far imply in particular that two Mori dream surfaces are isomorphic if and only if their Cox rings are isomorphic as graded rings. Moreover they prove the implication “(i) $\Rightarrow$ (ii)” of Theorem 2.37. For the other direction, one reduces finite generation of the Cox ring  $\mathcal{R}(X)$  to finite generation of the semiample subalgebras  $\oplus_{K \cap \lambda} \Gamma(X, \mathcal{R}_{[D]})$ , where  $\lambda^\circ \subseteq \text{Mov}(X)^\circ$ , which is given by classical results.

Recall that two Weil divisors  $D, D' \in \text{WDiv}(X)$  are said to be *Mori equivalent*, if there is a commutative diagram

$$\begin{array}{ccc}
 & X & \\
 \varphi(D) \swarrow & & \searrow \varphi(D') \\
 X(D) & \xleftarrow{\cong} & X(D')
 \end{array}$$

**Proposition 2.43.** *For any two Weil divisors  $D, D'$  on  $X$ , the following statements are equivalent.*

- (i) *The divisors  $D$  and  $D'$  are Mori equivalent.*
- (ii) *One has  $[D], [D'] \in \lambda^\circ$  for some GIT-chamber  $\lambda \in \Lambda(\overline{X}, H)$ .*

Let us have a look at the effect of blowing up and more general modifications on the Cox ring. In general, this is delicate, even finite generation may be lost. We discuss a class of toric ambient modifications having good properties in this regard; we restrict to hypersurfaces, for the general case see [19].

The starting point is a variety  $X_0$  arising from a bunched ring  $(R_0, \mathfrak{F}_0, \Phi_0)$  where the  $K_0$ -graded algebra  $R_0$  is of the form  $\mathbb{K}[T_1, \dots, T_r]/\langle f_0 \rangle$  and  $\mathfrak{F}_0$  consists of the (pairwise nonassociated  $K_0$ -prime) classes  $T_i + \langle f_0 \rangle$ ; as usual, we denote their  $K_0$ -degrees by  $w_i$ . We obtained the canonical toric embedding via

$$\begin{array}{ccccc}
 \overline{X}_0 & \supseteq & \widehat{X}_0 & \longrightarrow & \widehat{Z}_0 & \subseteq & \overline{Z}_0 \\
 \parallel H \downarrow & & & & \downarrow \parallel H & & \\
 X_0 & \longrightarrow & & & Z_0 & & 
 \end{array}$$

The morphism  $\widehat{Z}_0 \rightarrow Z_0$  is given by a map  $\widehat{\Sigma}_0 \rightarrow \Sigma_0$  of fans living in lattices  $F_0 = \mathbb{Z}^r$  and  $N_0$ . Let  $v_1, \dots, v_r$  be the primitive lattice vectors in the rays of  $\Sigma_0$  and suppose that for  $2 \leq d \leq r$ , the cone  $\sigma_0$  generated by  $v_1, \dots, v_d$  belongs to  $\Sigma_0$ . Consider the stellar subdivision  $\Sigma_1 \rightarrow \Sigma_0$  at a vector

$$v_\infty = a_1 v_1 + \dots + a_d v_d.$$

Let  $m_\infty$  be the index of this subdivision, i.e. the gcd of the entries of  $v_\infty$ , and denote the associated toric modification by  $\pi: Z_1 \rightarrow Z_0$ . Then we obtain the strict transform  $X_1 \subseteq Z_1$  mapping onto  $X_0 \subseteq Z_0$ . Moreover, we have commutative diagrams

$$\begin{array}{ccccc}
 & & \overline{Z}_1 & & \\
 & \swarrow \overline{\pi}_1 & & \searrow \overline{\pi} & \\
 \overline{Z}_1 & & \widehat{Z}_1 & & \overline{Z}_0 \\
 & \nwarrow \widehat{\pi}_1 & & \nearrow \widehat{\pi} & \\
 \widehat{Z}_1 & & & & \widehat{Z}_0 \\
 \downarrow p_1 / H_1 & & & & \downarrow / H_0 \quad p_0 \\
 Z_1 & \xrightarrow{\pi} & & & Z_0
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & \overline{Y}_1 & & \\
 & \swarrow \overline{\kappa}_1 & & \searrow \overline{\kappa} & \\
 \overline{X}_1 & & \widehat{Y}_1 & & \overline{X}_0 \\
 & \nwarrow \widehat{\kappa}_1 & & \nearrow \widehat{\kappa} & \\
 \widehat{X}_1 & & & & \widehat{X}_0 \\
 \downarrow p_1 / H_1 & & & & \downarrow / H_0 \quad p_0 \\
 X_1 & \xrightarrow{\kappa} & & & X_0
 \end{array}$$

where  $H_1 = \text{Spec}(\mathbb{K}[K_1])$  for  $K_1 = E_1/M_1$  in analogy to  $K_0 = E_0/M_0$  etc.. Note that the Cox ring  $\mathbb{K}[T_1, \dots, T_r, T_\infty]$  of  $\overline{Z}_1$  comes with a  $K_1$ -grading. Moreover, with respect to the coordinates corresponding to the rays of the fans  $\Sigma_i$ , the map  $\overline{\pi}: \overline{Z}_1 \rightarrow \overline{Z}_0$  is given by

$$\overline{\pi}(z_1, \dots, z_r, z_\infty) = (z_\infty^{a_1} z_1, \dots, z_\infty^{a_d} z_d, z_{d+1}, \dots, z_r).$$

We want to formulate an explicit condition on the setting which guarantees that  $\Gamma(\overline{X}_1, \mathcal{O})$  is the Cox ring of the proper transform  $X_1$ . For this, consider the grading

$$\mathbb{K}[T_1, \dots, T_r] = \bigoplus_{k \geq 0} \mathbb{K}[T_1, \dots, T_r]_k, \quad \text{where } \deg(T_i) := \begin{cases} a_i & i \leq d, \\ 0 & i \geq d+1. \end{cases}$$

Then we may write  $f_0 = g_{k_0} + \dots + g_{k_m}$  where  $k_0 < \dots < k_m$  and each  $g_{k_i}$  is a nontrivial polynomial having degree  $k_i$  with respect to this grading.

**Definition 2.44.** We say that the polynomial  $f_0 \in \mathbb{K}[T_1, \dots, T_r]$  is *admissible* if

- (i) the toric orbit  $0 \times \mathbb{T}^{r-d}$  intersects  $\overline{X}_0 = V(f_0)$ ,
- (ii)  $g_{k_0}$  is a  $K_1$ -prime polynomial in at least two variables.

Note that for the case of a free abelian group  $K_1$ , the second condition just means that  $g_{k_0}$  is an irreducible polynomial.

**Proposition 2.45.** *If, in the above setting, the polynomial  $f_0$  is admissible, then the Cox ring of the strict transform  $X_1 \subseteq Z_1$  is*

$$\mathcal{R}(X_1) = \mathbb{K}[T_1, \dots, T_r, T_\infty] / \langle f_1(T_1, \dots, T_r, {}^{m_\infty}\sqrt{T_\infty}) \rangle,$$

where in

$$f_1 := \frac{f_0(T_\infty^{a_1} T_1, \dots, T_\infty^{a_d} T_d, T_{d+1}, \dots, T_r)}{T_\infty^{k_0}} \in \mathbb{K}[T_1, \dots, T_r, T_\infty]$$

only powers  $T_\infty^{lm_\infty}$  with  $l \geq 0$  of  $T_\infty$  occur, and the notation  ${}^{m_\infty}\sqrt{T_\infty}$  means replacing  $T_\infty^{lm_\infty}$  with  $T_\infty^l$  in  $f_1$ .

**Example 2.46.** Consider the polynomial  $f_0 := T_1 + T_2^2 + T_3 T_4$  and the factorial algebra  $R_0 := \mathbb{K}[T_1, \dots, T_4]/f_0$ . Then  $R_0$  is graded by  $K_0 := \mathbb{Z}$  via the weight matrix

$$Q_0 = [2, 1, 1, 1].$$

With  $\mathfrak{F}_0 := (\overline{T}_1, \dots, \overline{T}_4)$  and  $\Phi_0 = \{\mathbb{Q}_{\geq 0}\}$ , we obtain a bunched ring  $(R_0, \mathfrak{F}_0, \Phi_0)$ . The associated variety  $X_0$  is a projective surface. In fact, there is an isomorphism  $X_0 \rightarrow \mathbb{P}_2$  induced by

$$\mathbb{K}^4 \rightarrow \mathbb{K}^3, \quad (z_1, z_2, z_3, z_4) \mapsto (z_2, z_3, z_4).$$

The canonical toric ambient variety  $Z_0$  of  $X_0$  is an open subset of the weighted projective space  $\mathbb{P}(2, 1, 1, 1)$ . To obtain a defining fan  $\Sigma_0$  of  $Z_0$ , consider the Gale dual matrix

$$P_0 := \begin{pmatrix} -1 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Let  $v_1, \dots, v_4$  denote its columns  $v_1, \dots, v_4$ . According to 2.24, the maximal cones of  $\Sigma_0$  are  $P(\gamma_0^\perp \cap \delta)$ , where  $\gamma_0$  runs through the covering collection  $\text{cov}(\Phi_0)$ . In terms of the columns  $v_1, \dots, v_4$  of  $P$  they are given as

$$\sigma_{1,2,3} := \text{cone}(v_1, v_2, v_3), \quad \sigma_{1,2,4} := \text{cone}(v_1, v_2, v_4), \quad \sigma_{3,4} := \text{cone}(e_3, e_4).$$

Now we subdivide  $\sigma_{1,2,3}$  by inserting the ray through  $v_\infty = 3v_1 + v_2 + 2v_3$ . Note that  $f_0$  is admissible; the  $g_{k_0}$ -term is  $T_2^2 + T_3T_4$ . According to 2.45, the defining equation of the Cox ring of the proper transform  $X_1$  is

$$f_1 = \frac{T_\infty^3 T_1 + T_\infty^2 T_2^2 + T_\infty^2 T_3 T_4}{T_\infty^2} = T_\infty T_1 + T_2^2 + T_3 T_4.$$

In order to obtain the grading matrix, we have to look at the matrix with the new primitive generators as columns. In abuse of notation, we put  $v_\infty = (-1, -1, -1)$  at the first place:

$$P_1 := \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The degree matrix is the Gale dual:

$$Q_1 = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix}.$$

Thus, up to renaming of the variables, we obtain the Cox ring of the singular del Pezzo surface considered in 2.25 and 2.36. In particular, we see that this del Pezzo surface is a modification of the projective plane.

### 3. THIRD LECTURE

**3.1. Varieties with torus action.** We describe the Cox ring of a variety with torus action following [22]. First recall the example of a complete toric variety  $X$ . Its Cox ring is given in terms of the prime divisors  $D_1, \dots, D_r$  in the boundary  $X \setminus T \cdot x_0$  of the open orbit:

$$\mathcal{R}(X) = \mathbb{K}[T^{D_1}, \dots, T^{D_r}], \quad \deg(T^{D_i}) = [D] \in \text{Cl}(X).$$

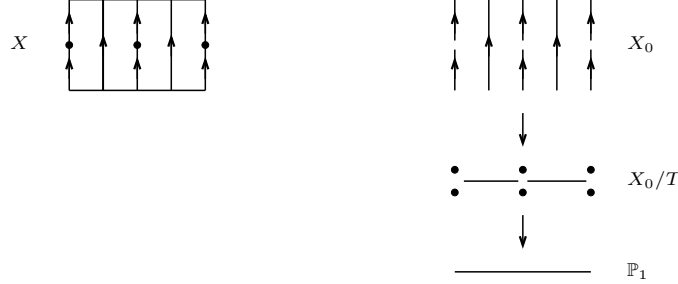
Now let  $X$  be any normal complete variety with finitely generated divisor class group and consider an effective algebraic torus action  $T \times X \rightarrow X$ , where  $\dim(T)$  may be less than  $\dim(X)$ . For a point  $x \in X$ , denote by  $T_x \subseteq T$  its isotropy group. The points with finite isotropy group form a non-empty  $T$ -invariant open subset

$$X_0 := \{x \in X; T_x \text{ is finite}\} \subseteq X.$$

This set will replace the open orbit of a toric variety. Let  $E_k$ , where  $1 \leq k \leq m$ , denote the prime divisors in  $X \setminus X_0$ ; note that each  $E_k$  is  $T$ -invariant with infinite generic isotropy, i.e. the subgroup of  $T$  acting trivially on  $E_k$  is infinite. According to a result of Sumihiro [40], there is a geometric quotient  $q: X_0 \rightarrow X_0/T$  with an irreducible normal but possibly non-separated orbit space  $X_0/T$ .



**Example 3.1.** Consider  $\mathbb{P}_1 \times \mathbb{P}_1$  the  $\mathbb{K}^*$ -action given w.r.t. inhomogeneous coordinates by  $t \cdot (z, w) = (z, tw)$ . Let  $X$  be the  $\mathbb{K}^*$ -equivariant blow up of  $\mathbb{P}_1 \times \mathbb{P}_1$  at the fixed points  $(0, 0)$ ,  $(1, 0)$  and  $(\infty, 0)$ .



The open set  $X_0$  is obtained by removing the two fixed point curves and the three isolated fixed points. The quotient  $X_0/T$  is the non-separated projective line with the points  $0, 1, \infty$  doubled; note that there is a canonical map  $X_0/T \rightarrow \mathbb{P}_1$ .

As it turns out, one always finds kind of *separation* for the orbit space in our setting: there are a rational map  $\pi: X_0/T \dashrightarrow Y$ , an open subset  $W \subseteq X_0/T$  and prime divisors  $C_0, \dots, C_r$  on  $Y$  such that following holds:

- the complement of  $W$  in  $X_0/T$  is of codimension at least two and the restriction  $\pi: W \rightarrow Y$  is a local isomorphism,
- each inverse image  $\pi^{-1}(C_i)$  is a disjoint union of prime divisors  $C_{ij}$ , where  $1 \leq j \leq n_i$ ,
- the map  $\pi$  is an isomorphism over  $Y \setminus (C_0 \cup \dots \cup C_r)$  and all prime divisors of  $X_0$  with nontrivial generic isotropy occur among the  $D_{ij} := q^{-1}(C_{ij})$ .

Let  $l_{ij} \in \mathbb{Z}_{\geq 1}$  be the order of the generic isotropy group of the  $T$ -action on the prime divisor  $D_{ij}$  and define monomials  $T_i^{l_i} := T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}}$  in the variables  $T_{ij}$ . Moreover, let  $1_{E_k}$  and  $1_{D_{ij}}$  denote the canonical sections of  $E_k$  and  $D_{ij}$ .

**Theorem 3.2.** *There is a graded injection  $\mathcal{R}(Y) \rightarrow \mathcal{R}(X)$  of Cox rings and  $S_k \mapsto 1_{E_k}$ ,  $T_{ij} \mapsto 1_{D_{ij}}$  defines an isomorphism of  $\text{Cl}(X)$ -graded rings*

$$\mathcal{R}(X) \cong \mathcal{R}(Y)[T_{ij}, S_1, \dots, S_m; 0 \leq i \leq r, 1 \leq j \leq n_i] / \langle T_i^{l_i} - 1_{C_i}; 0 \leq i \leq r \rangle,$$

where the  $\text{Cl}(X)$ -grading on the right hand side is defined by associating to  $S_k$  the class of  $E_k$  and to  $T_{ij}$  the class of  $D_{ij}$ .

As a direct consequence, we obtain that finite generation of the Cox ring of  $X$  is determined by the separation  $Y$  of the orbit space  $X_0/T$ .

**Corollary 3.3.** *The Cox ring  $\mathcal{R}(X)$  is finitely generated if and only if  $\mathcal{R}(Y)$  is so.*

We specialize to the case that the  $T$ -action on  $X$  is of *complexity* one, i.e. its biggest orbits are of codimension one in  $X$ . Then the orbit space  $X_0/T$  is of dimension one and smooth.

**Remark 3.4.** For a normal complete variety  $X$  with a torus action  $T \times X \rightarrow X$  of complexity one, the following statements are equivalent.

- $\text{Cl}(X)$  is finitely generated.
- $X$  is rational.

Moreover, if one of these statements holds, then the separation of the orbit space is a morphism  $\pi: X_0/T \rightarrow \mathbb{P}_1$ .

The former prime divisors  $C_i \subseteq Y$  are now points  $a_0, \dots, a_r \in \mathbb{P}_1$  and  $\pi^{-1}(a_i)$  consists of points  $x_{i1}, \dots, x_{in_i} \in X_0/T$ . As before, we may assume that all prime divisors of  $X_0$  with non trivial generic isotropy occur among the prime divisors

$D_{ij} = q^{-1}(x_{ij})$ . Consider again the monomials  $T_i^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ , where  $l_{ij} \in \mathbb{Z}_{\geq 1}$  is the order of the generic isotropy group of  $D_{ij}$ . Write  $a_i = [b_i, c_i]$  with  $b_i, c_i \in \mathbb{K}$ . For every  $0 \leq i \leq r-2$  set  $j := i+1$ ,  $k := i+2$  and define a trinomial

$$g_i := (b_j c_k - c_j b_k) T_i^{l_i} + (b_k c_i - c_k b_i) T_j^{l_j} + (b_i c_j - c_i b_j) T_k^{l_k}.$$

**Theorem 3.5.** *Let  $X$  be a normal complete variety with finitely generated divisor class group and an effective algebraic torus action  $T \times X \rightarrow X$  of complexity one. Then, in terms of the data defined above, the Cox ring of  $X$  is given as*

$$\mathcal{R}(X) \cong \mathbb{K}[S_1, \dots, S_m, T_{ij}; 0 \leq i \leq r, 1 \leq j \leq n_i] / \langle g_i; 0 \leq i \leq r-2 \rangle.$$

where  $1_{E_k}$  corresponds to  $S_k$ , and  $1_{D_{ij}}$  to  $T_{ij}$ , and the  $\text{Cl}(X)$ -grading on the right hand side is defined by associating to  $S_k$  the class of  $E_k$  and to  $T_{ij}$  the class of  $D_{ij}$ .

Let us discuss some applications of Theorem 3.5. In a first application we compute the Cox ring of a surface obtained by repeated blowing up points of the projective plane that lie on a given line; the case  $n_0 = \dots = n_r = 1$  was done by other methods in [36].

**Example 3.6** (Blowing up points on a line). Consider a line  $Y \subseteq \mathbb{P}_2$  and points  $p_0, \dots, p_r \in Y$ . Let  $X$  be the surface obtained by blowing up  $n_i$  times the point  $p_i$ , where  $0 \leq i \leq r$ ; in every step, we identify  $Y$  with its proper transform and  $p_i$  with the point in the intersection of  $Y$  with the exceptional curve. Set

$$g_i := (b_j c_k - c_j b_k) T_i + (b_k c_i - c_k b_i) T_j + (b_i c_j - c_i b_j) T_k,$$

with  $p_i = [b_i, c_i]$  in  $Y = \mathbb{P}_1$ , the monomials  $T_i = T_{i0} \cdots T_{in_i}$  and the indices  $k = i+2$ ,  $j = i+1$ . Then the Cox ring of the surface  $X$  is given as

$$\mathcal{R}(X) = \mathbb{K}[T_{ij}, S; 0 \leq i \leq r, 0 \leq j \leq n_i] / \langle g_i; 0 \leq i \leq r-2 \rangle.$$

We verify this using a  $\mathbb{K}^*$ -action; note that blowing up  $\mathbb{K}^*$ -fixed points always can be made equivariant. With respect to suitable homogeneous coordinates  $z_0, z_1, z_2$ , we have  $Y = V(z_0)$ . Let  $\mathbb{K}^*$  act via

$$t \cdot [z] := [z_0, tz_1, tz_2].$$

Then  $E_1 := Y$  is a fixed point curve, the  $j$ -th (equivariant) blowing up of  $p_i$  produces an invariant exceptional divisor  $D_{ij}$  with a free  $\mathbb{K}^*$ -orbit inside and Theorem 3.5 gives the claim.

As a further application we indicate a general recipe for computing the Cox ring of a rational hypersurface given by a trinomial equation in the projective space. We perform this for the  $E_6$  cubic surface in  $\mathbb{P}_3$ ; the Cox ring of the resolution of this surface has been computed in [23].

**Example 3.7** (The  $E_6$  cubic surface). There is a cubic surface  $X$  in the projective space having singular locus  $X^{\text{sing}} = \{x_0\}$  and  $x_0$  of type  $E_6$ . The surface is unique up to projectivity and can be realized as follows:

$$X = V(z_1 z_2^2 + z_2 z_0^2 + z_3^3) \subseteq \mathbb{P}_3.$$

Note that the defining equation is a trinomial but not of the shape of those occurring in Theorem 3.5. However, any trinomial hypersurface in a projective space comes with a complexity one torus action. Here, we have the  $\mathbb{K}^*$ -action

$$t \cdot [z_0, \dots, z_4] = [z_0, t^{-3} z_1, t^3 z_2, t z_3].$$

This allows us to use Theorem 3.5 for computing the Cox ring. The task is to find the divisors  $E_k, D_{ij}$  and the orders  $l_{ij}$  of the isotropy groups. Note that  $\mathbb{K}^*$  acts freely on the big torus of  $\mathbb{P}_3$ . The intersections of  $X$  with the toric prime divisors  $V(z_i) \subseteq \mathbb{P}_3$  are given as

$$X \cap V(z_0) = V(z_0, z_1 z_2^2 + z_3^3), \quad X \cap V(z_1) = V(z_1, z_2 z_0^2 + z_3^3),$$

$$X \cap V(z_3) = V(z_3, z_2(z_1 z_2 + z_0^2)) = (X \cap V(z_2)) \cup (X \cap V(z_1 z_2 + z_0^2)).$$

The first two sets are irreducible and both of them intersect the big torus orbit of the respective toric prime divisors  $V(z_0)$  and  $V(z_1)$ . In order to achieve this also for  $V(z_2)$ ,  $V(z_3)$ , we use a suitable weighted blow up of  $\mathbb{P}_3$  at  $V(z_2) \cap V(z_3)$ . In terms of fans, this means to perform a certain stellar subdivision. Consider the matrices

$$P = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad P' = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The columns  $v_0, \dots, v_3$  of  $P$  are the primitive generators of the fan  $\Sigma$  of  $Z := \mathbb{P}_3$  and we obtain a fan  $\Sigma'$  subdividing  $\Sigma$  at the last column  $v_4$  of  $P'$ ; note that  $v_4$  is located on the tropical variety  $\text{trop}(X)$ . Consider the associated toric morphism and the proper transform

$$\pi: Z \rightarrow Z, \quad X' := \overline{\pi^{-1}(X \cap \mathbb{T}^3)} \subseteq Z_1.$$

A simple computation shows that the intersection of  $X'$  with the toric prime divisors of  $Z'$  is irreducible and intersects their big orbits. Moreover,  $\pi: X' \rightarrow X$  is an isomorphism, because along  $X'$  nothing gets contracted. To proceed, note that we have no divisors of type  $E_k$  and that there is a commutative diagram

$$\begin{array}{ccc} X'_0 & \longrightarrow & Z'_0 \\ \downarrow / \mathbb{K}^* & & \downarrow / \mathbb{K}^* \\ X'_0 / \mathbb{K}^* & \longrightarrow & Z'_0 / \mathbb{K}^* \end{array}$$

We determine the quotient  $Z'_0 \rightarrow Z'_0 / \mathbb{K}^*$ . The group  $\mathbb{K}^*$  acts on  $Z'$  via homomorphism  $\lambda_v: \mathbb{K}^* \rightarrow \mathbb{T}^3$  corresponding to  $v = (-3, 3, 1) \in \mathbb{Z}^3$ . The quotient by this action is the toric morphism given by any map  $S: \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$  having  $\mathbb{Z} \cdot v$  as its kernel. We take  $S$  as follows and compute the images of the columns  $v_0, \dots, v_4$  of  $P'$ :

$$S := \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \end{pmatrix}, \quad S \cdot P' = \begin{pmatrix} -4 & 1 & 0 & 3 & 3 \\ 2 & 0 & 1 & -3 & 0 \end{pmatrix}.$$

This shows that the toric divisors  $D_Z^1$  and  $D_Z^4$  corresponding to  $v_1$  and  $v_4$  are mapped to a doubled divisor in the non-separated quotient; see also [1]. The generic isotropy group of the  $\mathbb{K}^*$ -action along the toric divisor  $D_Z^i$  is given as the gcd  $l_i$  of the entries of the  $i$ -th column of  $S \cdot P'$ . We obtain

$$l_0 = 2, \quad l_1 = 1, \quad l_2 = 1, \quad l_3 = 3, \quad l_4 = 3.$$

By construction, the divisors  $D_X^i := D_Z^i$  of the embedded variety  $X'_0 \subseteq Z'_0$  inherit the orders  $l_i$  of the isotropy groups and the behaviour with respect to the quotient map  $X'_0 \rightarrow X'_0 / \mathbb{K}^*$ . Renaming these divisors by

$$D_{01} := D_X^1, \quad D_{02} := D_X^4, \quad D_{11} := D_X^3, \quad D_{21} := D_X^0,$$

we arrive in the setting of Theorem 3.5; since  $D_X^2$  has trivial isotropy group and gets not multiplied by the quotient map, it does not occur here. The Cox ring of  $X \cong X'$  is then given as

$$\mathcal{R}(X) \cong \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] / \langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2 \rangle.$$

**3.2. Writing down all Cox rings.** As we observed, the Cox ring of a rational complete normal variety with a complexity one torus action admits a nice presentation by trinomial relations. Now we ask which of these trinomial rings occur as a Cox ring. First, we formulate the answer in algebraic terms and then turn to the geometric point of view; for the details, we refer to [20].

**Construction 3.8.** Fix  $r \in \mathbb{Z}_{\geq 1}$ , a sequence  $n_0, \dots, n_r \in \mathbb{Z}_{\geq 1}$ , set  $n := n_0 + \dots + n_r$ , and fix integers  $m \in \mathbb{Z}_{\geq 0}$  and  $0 < s < n + m - r$ . The input data are

- a sequence  $A = (a_0, \dots, a_r)$  of vectors  $a_i = (b_i, c_i)$  in  $\mathbb{K}^2$  such that any pair  $(a_i, a_j)$  with  $j \neq i$  is linearly independent,
- an integral block matrix  $P$  of size  $(r + s) \times (n + m)$  the columns of which are pairwise different primitive vectors generating  $\mathbb{Q}^{r+s}$  as a cone:

$$P = \begin{pmatrix} P_0 & 0 \\ d & d' \end{pmatrix},$$

where  $d$  is an  $(s \times n)$ -matrix,  $d'$  an  $(s \times m)$ -matrix and  $P_0$  an  $(r \times n)$ -matrix build from tuples  $l_i := (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}_{\geq 1}^{n_i}$  in the following way

$$P_0 = \begin{pmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{pmatrix}.$$

Now we associate to any such pair  $(A, P)$  a ring, graded by  $K := \mathbb{Z}^{n+m}/\text{im}(P^*)$ , where  $P^*$  is the transpose of  $P$ . For every  $0 \leq i \leq r$ , define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}}.$$

Moreover, for any two indices  $0 \leq i, j \leq r$ , set  $\alpha_{ij} := \det(a_i, a_j) = b_i c_j - b_j c_i$  and for any three indices  $0 \leq i < j < k \leq r$  define a trinomial

$$g_{i,j,k} := \alpha_{jk} T_i^{l_i} + \alpha_{ki} T_j^{l_j} + \alpha_{ij} T_k^{l_k}.$$

Note that all trinomials  $g_{i,j,k}$  are  $K$ -homogeneous of the same degree. Setting  $g_i := g_{i,i+1,i+2}$ , we obtain a  $K$ -graded factor algebra

$$R(A, P) := \mathbb{K}[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m] / \langle g_i; 0 \leq i \leq r-2 \rangle.$$

**Remark 3.9.** The polynomials  $g_{i,j,k}$  can be written as determinants in the following way:

$$g_{i,j,k} = \det \begin{pmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \\ T_i^{l_i} & T_j^{l_j} & T_k^{l_k} \end{pmatrix}.$$

**Theorem 3.10.** Let  $(A, P)$  be data as in 3.8. Then the algebra  $R := R(A, P)$  is a normal factorially  $K$ -graded complete intersection; we have  $R^* = \mathbb{K}^*$ , the  $K$ -grading is almost free and  $R_0 = \mathbb{K}$  holds. Moreover, the variables  $T_{ij}, S_k$  define a system of pairwise nonassociated  $K$ -prime generators for  $R$ .

We say that the pair  $(A, P)$  is *sincere*, if  $r \geq 2$  and  $n_i l_{ij} > 1$  for all  $i, j$  hold; this ensures that there exist in fact relations  $g_{i,j,k}$  and none of these relations contains a linear term. The following statement tells us which of the  $R(A, P)$  are factorial.

**Theorem 3.11.** Let  $(A, P)$  be a sincere pair. Then the following statements are equivalent.

- The algebra  $R(A, P)$  is a unique factorization domain.
- The group  $\mathbb{Z}^r / \text{im}(P_0)$  is torsion free.
- The numbers  $\gcd(l_i)$  and  $\gcd(l_j)$  are coprime for any  $0 \leq i < j \leq r$ .

More generally, one can in a similar manner to 3.10 determine all affine algebras  $R$  with an effective factorial  $K$ -grading of complexity one such that  $R_0 = \mathbb{K}$  holds, see [20]. The characterization of the factorial ones is the same as in 3.11. In dimensions two and three, the factorial algebras with a complexity one multigrading are described in early work of Mori [30] and Ishida [27].

**Example 3.12.** The algebra  $\mathbb{K}[T_{01}, T_{11}, T_{21}]/\langle T_{01}^2 + T_{11}^2 + T_{21}^2 \rangle$  becomes graded by the group  $K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  via

$$\deg(T_{01}) = (1, \bar{0}, \bar{0}), \quad \deg(T_{11}) = (1, \bar{1}, \bar{0}), \quad \deg(T_{21}) = (1, \bar{0}, \bar{1}).$$

This is an effective factorial grading of complexity one. However, the grading is not almost free. Thus, the algebra is not a Cox ring.

Let us turn to geometric aspects. We want to see the effective complexity one torus action on the varieties having a Cox ring  $R(A, P)$ . Existence of this action can be obtained by looking at the maximal possible grading of  $R(A, P)$ . We use here the canonical toric embedding which provides a little more geometric information.

**Construction 3.13.** Take a  $K$ -graded algebra  $R = R(A, P)$  as constructed in 3.8 and let  $\mathfrak{F}$  be the system of pairwise nonassociated  $K$ -prime generators defined by the variables  $T_{ij}$  and  $S_k$ . Given any  $\mathfrak{F}$ -bunch  $\Phi$ , we obtain a bunched ring  $(R, \mathfrak{F}, \Phi)$  and an associated variety  $X$ . Consider the mutually dual sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & F & \xrightarrow{P} & N, \\ & & & & & & \\ 0 & \longleftarrow & K & \xleftarrow{Q} & E & \xleftarrow{P^*} & M \longleftarrow 0. \end{array}$$

Recall that  $Q: E \rightarrow K$  defines the  $K$ -degrees of the variables  $T_{ij}$ ,  $S_k$  and note that  $P$  is indeed our defining matrix of the algebra  $R(A, P)$ . Now, let  $\widehat{\Sigma}$  denote the fan in  $F$  generated by  $\Phi$  and let  $\Sigma$  be its quotient fan in  $N$ ; its rays have the columns  $v_{ij}$  and  $v_k$  of  $P$  as their primitive generators. Moreover, consider

$$\overline{X} := V(g_0, \dots, g_{r-2}) \subseteq \mathbb{K}^{n+m}.$$

The fan  $\widehat{\Sigma}$  defines an open toric subvariety  $\widehat{Z} \subseteq \mathbb{K}^{n+m}$  and the toric morphism  $p: \widehat{Z} \rightarrow Z$  defined by  $P: F \rightarrow N$  onto the toric variety  $Z$  associated to  $\Sigma$  is a characteristic space. The canonical toric embedding of  $X$  was obtained via the commutative diagram

$$\begin{array}{ccccc} \overline{X} & \supseteq & \widehat{X} & \longrightarrow & \widehat{Z} & \subseteq & \mathbb{K}^{n+m} \\ & & \parallel^H \downarrow & & \downarrow \parallel^H & & \\ & & X & \longrightarrow & Z & & \end{array}$$

Let  $T_Z := \text{Spec } \mathbb{K}[M]$  be the acting torus of  $Z$  and let  $T \subseteq T_Z$  be the subtorus corresponding to the inclusion  $0 \times \mathbb{Z}^s \rightarrow N$ . Then  $T$  acts on  $Z$  leaving  $X \subseteq Z$  invariant and  $T \times X \rightarrow X$  is an effective complexity one action. We also write  $X = X(A, P, \Phi)$  for this  $T$ -variety.

**Theorem 3.14.** *Let  $X$  be a normal complete  $A_2$ -variety with an effective complexity one torus action. Then  $X$  is equivariantly isomorphic to a variety  $X(A, P, \Phi)$  constructed in 3.13*

In particular, this enables us to apply the language of bunched rings to varieties with a complexity one torus action. The following is an application of 2.31 and 2.34.

**Proposition 3.15.** *Let  $X$  be a complete normal rational variety with an effective algebraic torus action  $T \times X \rightarrow X$  of complexity one.*

(i) *The cone of divisor classes without fixed components is given by*

$$\bigcap_{1 \leq k \leq m} \text{cone}([E_s], [D_{ij}]; s \neq k) \cap \bigcap_{\substack{0 \leq i \leq r \\ 1 \leq j \leq n_i}} \text{cone}([E_k], [D_{st}]; (s, t) \neq (i, j)).$$

(ii) For any  $0 \leq i \leq r$ , one obtains a canonical divisor for  $X$  by

$$\max(0, r-1) \cdot \sum_{j=0}^{n_i} l_{ij} D_{ij} - \sum_{k=1}^m E_k - \sum_{i,j} D_{ij}.$$

The explicit description of rational varieties with a complexity one torus action by the data  $(A, P)$  can be used for classifications. We will present a result of [21] on Fano threefolds with free class group of rank one; we first note a general observation on varieties of this type made there.

Let  $X$  be an arbitrary complete  $d$ -dimensional variety with divisor class group  $\text{Cl}(X) \cong \mathbb{Z}$ ; thus, we do not require the presence of a torus action. The Cox ring  $\mathcal{R}(X)$  is finitely generated and the total coordinate space  $\overline{X} := \text{Spec } \mathcal{R}(X)$  is a factorial affine variety coming with an action of  $\mathbb{K}^*$  defined by the  $\text{Cl}(X)$ -grading of  $\mathcal{R}(X)$ . Choose a system  $f_1, \dots, f_\nu$  of homogeneous pairwise nonassociated prime generators for  $\mathcal{R}(X)$ . This provides a  $\mathbb{K}^*$ -equivariant embedding

$$\overline{X} \rightarrow \mathbb{K}^\nu, \quad \bar{x} \mapsto (f_1(\bar{x}), \dots, f_\nu(\bar{x})),$$

where  $\mathbb{K}^*$  acts diagonally with the weights  $w_i = \deg(f_i) \in \text{Cl}(X) \cong \mathbb{Z}$  on  $\mathbb{K}^\nu$ . Moreover,  $X$  is the geometric  $\mathbb{K}^*$ -quotient of  $\widehat{X} := \overline{X} \setminus \{0\}$ , and the quotient map  $p: \widehat{X} \rightarrow X$  is a characteristic space.

**Proposition 3.16.** *For any  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_\nu) \in \widehat{X}$  the local divisor class group  $\text{Cl}(X, x)$  of  $x := p(\bar{x})$  is finite of order  $\gcd(w_i; \bar{x}_i \neq 0)$ . The index of the Picard group  $\text{Pic}(X)$  in  $\text{Cl}(X)$  is given by*

$$[\text{Cl}(X) : \text{Pic}(X)] = \text{lcm}_{x \in X} (|\text{Cl}(X, x)|).$$

Suppose that the ideal of  $\overline{X} \subseteq \mathbb{K}^\nu$  is generated by  $\text{Cl}(X)$ -homogeneous polynomials  $g_1, \dots, g_{\nu-d-1}$  of degree  $\gamma_j := \deg(g_j)$ . Then one obtains

$$-\mathcal{K}_X = \sum_{i=1}^\nu w_i - \sum_{j=1}^{\nu-d-1} \gamma_j, \quad (-\mathcal{K}_X)^d = \left( \sum_{i=1}^\nu w_i - \sum_{j=1}^{\nu-d-1} \gamma_j \right)^d \frac{\gamma_1 \cdots \gamma_{\nu-d-1}}{w_1 \cdots w_\nu}$$

for the anticanonical class  $-\mathcal{K}_X \in \text{Cl}(X) \cong \mathbb{Z}$ . In particular,  $X$  is a Fano variety if and only if the following inequality holds

$$\sum_{j=1}^{\nu-d-1} \gamma_j < \sum_{i=1}^\nu w_i.$$

Combining this with the explicit description in the presence of a complexity one torus action, one obtains bounds for the defining data. This leads to classification results. We present here the results for locally factorial threefolds; see [21] for details and more.

**Theorem 3.17.** *The following table lists the Cox rings  $\mathcal{R}(X)$  of the three-dimensional locally factorial non-toric Fano varieties  $X$  with an effective two torus action and  $\text{Cl}(X) = \mathbb{Z}$ .*

No.	$\mathcal{R}(X)$	$(w_1, \dots, w_5)$	$(-K_X)^3$
1	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^5 + T_3^3 + T_4^2 \rangle$	$(1, 1, 2, 3, 1)$	8
2	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 T_3^4 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
3	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^2 T_3^3 + T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 2, 3)$	8
4	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle$	$(1, 1, 1, 1, 1)$	54
5	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^2 + T_3 T_4^2 + T_5^3 \rangle$	$(1, 1, 1, 1, 1)$	24
6	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^3 + T_3 T_4^3 + T_5^4 \rangle$	$(1, 1, 1, 1, 1)$	4

7	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^3 + T_3 T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 2)$	16
8	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^5 + T_3 T_4^5 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2
9	$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^5 + T_3 T_4^3 + T_5^2 \rangle$	$(1, 1, 1, 1, 3)$	2

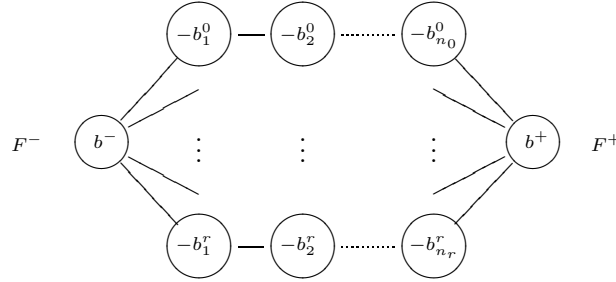
**3.3.  $\mathbb{K}^*$ -surfaces.** Here we take a closer look at the first non-trivial examples of complexity one torus actions.  $\mathbb{K}^*$ -surfaces are studied by many authors; a classical reference is the work of Orlik and Wagreich [37]. Recall that a fixed point of a  $\mathbb{K}^*$ -surface is said to be

- *elliptic* if it is isolated and lies in the closure of infinitely many  $\mathbb{K}^*$ -orbits,
- *parabolic* if it belongs to a fixed point curve,
- *hyperbolic* if it is isolated and lies in the closure of two  $\mathbb{K}^*$ -orbits.

These are in fact the only possible types of fixed points for a normal  $\mathbb{K}^*$ -surface. For normal projective  $\mathbb{K}^*$ -surfaces  $X$ , there is always a *source*  $F^+ \subseteq X$  and a *sink*  $F^- \subseteq X$ . They are characterized by the behaviour of general points: there is a non-empty open set  $U \subseteq X$  with

$$\lim_{t \rightarrow 0} t \cdot x \in F^+, \quad \lim_{t \rightarrow \infty} t \cdot x \in F^- \quad \text{for all } x \in U.$$

The source can either consist of an elliptic fixed point or it is a curve of parabolic fixed points; the same holds for the sink. Any fixed point outside source or sink is hyperbolic. Note that in Example 3.1, we have two curves of parabolic fixed points and three hyperbolic fixed points. To every smooth projective  $\mathbb{K}^*$ -surface  $X$  having no elliptic fixed points Orlik and Wagreich associated a graph of the following shape:



The two fixed point curves of  $X$  occur as  $F^-$  and  $F^+$  in the graph. The other vertices represent the invariant irreducible contractible curves  $D_{ij} \subseteq X$  different from  $F^-$  and  $F^+$ . The label  $-b_j^i$  is the self intersection number of  $D_{ij}$ , and two of the  $D_{ij}$  are joined by an edge if and only if they have a common (fixed) point. Every  $D_{ij}$  is the closure of a non-trivial  $\mathbb{K}^*$ -orbit.

We show how to read off the Cox ring of a rational  $X$  from its Orlik-Wagreich graph. By [37, Sec. 3.5], the order  $l_{ij}$  of generic isotropy group of  $D_{ij}$  equals the numerator of the canceled continued fraction

$$b_1^i - \frac{1}{b_2^i - \frac{1}{\dots - \frac{1}{b_{j-1}^i}}}$$

Moreover, there is a canonical isomorphism  $\mathbb{P}_1 = Y = F^-$  identifying  $a_i \in Y$  with the point in  $F^- \cap D_{i1}$ . In the terminology introduced before, we thus obtain the following.

**Theorem 3.18.** *Let  $X$  be a smooth complete rational  $\mathbb{K}^*$ -surface without elliptic fixed points. Then the assignments  $S^\pm \mapsto 1_{F^\pm}$  and  $T_{ij} \mapsto 1_{D_{ij}}$  define an isomorphism*

$$\mathcal{R}(X) \cong \mathbb{K}[S^+, S^-, T_{ij}; 0 \leq i \leq r, 1 \leq j \leq n_i] / \langle g_i; 0 \leq i \leq r-2 \rangle$$

of  $\text{Cl}(X)$ -graded rings, where the  $\text{Cl}(X)$ -grading on the right hand side is defined by associating to  $S^\pm$  the class of  $F^\pm$  and to  $T_{ij}$  the class of  $D_{ij}$ .

The Cox ring of a general rational  $\mathbb{K}^*$ -surface  $X$  is then computed as follows. Blowing up the (possible) elliptic fixed points suitably often gives a surface with two parabolic fixed point curves. Resolving the remaining singularities, we arrive at a smooth  $\mathbb{K}^*$ -surface  $\tilde{X}$  without parabolic fixed points, the so called *canonical resolution* of  $X$ ; note that this is in general not the minimal resolution. The Cox ring  $\mathcal{R}(\tilde{X})$  is computed as above. For the Cox ring  $\mathcal{R}(X)$ , we need the divisors of the type  $E_k$  and  $D_{ij}$  in  $X$  and the orders  $l_{ij}$  of the generic isotropy groups of the  $D_{ij}$ . Each of these divisors is the image of a non-exceptional divisor of the same type in  $\tilde{X}$ ; to see this for the  $D_{ij}$ , note that  $X_0$  is the open subset of  $\tilde{X}_0$  obtained by removing the exceptional locus of  $\tilde{X} \rightarrow X$  and thus  $X_0/\mathbb{K}^*$  is an open subset of  $\tilde{X}_0/\mathbb{K}^*$ . Moreover, by equivariance, the orders  $l_{ij}$  in  $X$  are the same as in  $\tilde{X}$ . Consequently, the Cox ring  $\mathcal{R}(X)$  is obtained from  $\mathcal{R}(\tilde{X})$  by removing those generators that correspond to the exceptional curves arising from the resolution.

Now let us look at rational normal complete  $\mathbb{K}^*$ -surfaces  $X$  with the methods presented in the preceding subsection. First recall that by finite generation of its Cox ring,  $X$  is  $\mathbb{Q}$ -factorial and projective. Moreover,  $X$  arises from a ring  $R(A, P)$  as in Construction 3.13. As any surface,  $X$  is the only representative of its small birational class which means that no  $\mathfrak{F}$ -bunch  $\Phi$  needs to be specified in the defining data.

**Remark 3.19.** Let  $(A, P)$  be data as in 3.8. In order that  $R(A, P)$  defines a surface  $X$ , we necessarily have  $s = 1$  and for  $m$  we have the following three possibilities

- $m = 0$ . This holds if and only if  $X$  has two elliptic fixed points. In this case  $d'$  is empty.
- $m = 1$ . This holds if and only if  $X$  has one fixed point curve and one elliptic fixed point. In this case, we may assume  $d' = (-1)$ .
- $m = 2$ . This holds if and only if  $X$  has two elliptic fixed points. In this case, we may assume  $d' = (1, -1)$ .

In the fan  $\Sigma$  of the canonical ambient toric variety  $Z$  of  $X$ , the parabolic fixed point curves correspond to rays through  $(0, \dots, 0, \pm 1)$ , the elliptic fixed points to full dimensional cones  $\sigma^\pm$  and the hyperbolic fixed points to two-dimensional cones.

We demonstrate the concrete working with  $\mathbb{K}^*$ -surfaces by resolving singularities in two examples; for a detailed general treatment we refer to [24]. Similar to the canonical resolution of Orlik and Wagreich, the first step is resolving singular elliptic fixed points by inserting rays through  $(0, \dots, 0, \pm 1)$ . The remaining singularities are then resolved by regular subdivision of two-dimensional singular cones of  $\Sigma$ . The first example is our surface 2.25.

**Example 3.20.** In the setting of 3.8, let  $r = 2$ , set  $n_0 = 2$ ,  $n_1 = 1$ ,  $n_2 = 2$  and  $m = 0$ . For  $A$  choose the vectors  $(-1, 0)$ ,  $(1, -1)$  and  $(0, 1)$ . Consider the matrix

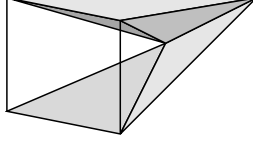
$$P = \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix}.$$



Then the defining equation of  $\overline{X} \subseteq \mathbb{K}^5$  is  $g = T_{01}T_{02} + T_{11}^2 + T_{21}T_{22}$ . Observe that the Gale dual matrix is

$$Q = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix}.$$

Thus,  $X$  is the del Pezzo surface we already encountered in 2.25. Consider the canonical toric ambient variety  $Z$ . Its fan  $\Sigma$  looks as follows:



More precisely, in terms of the columns  $v_{01}, v_{02}, v_{11}, v_{21}, v_{22}$  of  $P$ , the maximal cones of  $\Sigma$  are

$$\sigma^- := \text{cone}(v_{01}, v_{11}, v_{21}), \quad \sigma^+ := \text{cone}(v_{02}, v_{12}, v_{22}),$$

$$\tau_{012} := \text{cone}(v_{01}, v_{02}), \quad \tau_{212} := \text{cone}(v_{21}, v_{22}).$$

As we already observed,  $X$  comes with one singularity  $x_0 \in X$ ; it is the toric fixed point corresponding to the cone  $\sigma^-$ . Set

$$v_1 := (0, 0, -1), \quad v_{10} := (1, 0, 0).$$

Inserting these vectors as columns at the right places of the matrix  $P$  gives the describing matrix for the resolution  $\tilde{X}$  of  $X$ :

$$\tilde{P} = \begin{pmatrix} -1 & -1 & 1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}.$$

This allows us to read off the defining relation of the Cox ring; the degrees of the generators are the columns of the Gale dual matrix  $\tilde{Q}$ . Concretely, we obtain:

$$\mathcal{R}(X) = \mathbb{K}[T_{ij}, S; i = 0, 1, 2 \ j = 1, 2] / \langle T_{01}T_{02} + T_{11}T_{12}^2 + T_{21}T_{22} \rangle,$$

$$\tilde{Q} := \begin{pmatrix} 0 & 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

The self intersection numbers of the curves corresponding to  $v_1$  and  $v_{10}$  are computed as in 2.35 and both equal  $-2$ . Thus, the singularity of  $X$  is of type  $A_2$ .

In the second example, we compute the Cox ring of the minimal resolution of the  $E_6$  cubic surface from 3.7; the result was first obtained by Hassett and Tschinkel, without using the  $\mathbb{K}^*$ -action [23, Theorem 3.8].

**Example 3.21.** In the setting of 3.8, let  $r = 2$ , set  $n_0 = 2, n_1 = 1, n_2 = 1$  and  $m = 0$ . For  $A$  choose the vectors  $(-1, 0), (1, -1)$  and  $(0, 1)$ . Consider the matrix

$$P = \begin{pmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{pmatrix}.$$

The resulting surface  $X$  is the  $E_6$  cubic surface 3.7. Its Cox ring and degree matrix are given as

$$\mathcal{R}(X) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] / \langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2 \rangle$$

$$Q := \begin{pmatrix} 3 & 1 & 2 & 3 \end{pmatrix}.$$

Let us look at the fan of the toric ambient variety. In terms of the columns  $v_{01}, v_{02}, v_{11}, v_{21}$  of  $P$ , its maximal cones are

$$\sigma^- := \text{cone}(v_{01}, v_{11}, v_{21}), \quad \sigma^+ := \text{cone}(v_{02}, v_{11}, v_{22}), \quad \tau_{012} := \text{cone}(v_{01}, v_{02}).$$

The toric fixed point corresponding to  $\sigma^+$  is the singularity  $x_0 \in X$ . It is singular for two reasons: firstly  $\sigma^+$  is singular, secondly the total coordinate space

$$\overline{X} = V(T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2)$$

is singular along the fiber  $\mathbb{K}^* \times 0 \times 0 \times 0$  over  $x_0$ . Subdividing along  $(0, 0, 1)$  and further resolving gives

$$\tilde{P} = \begin{pmatrix} -1 & -3 & -2 & -1 & 3 & 2 & 1 & 0 & 0 & 0 \\ -1 & -3 & -2 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -1 & -2 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note that inserting first  $(0, 0, 1)$  is part of the first step of the canonical resolution. The Cox ring of the resolution and its degree matrix are thus given by

$$\mathcal{R}(\tilde{X}) = \frac{\mathbb{K}[T_{01}, T_{02}, T_{03}, T_{04}, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, S]}{\langle T_{01}T_{02}^3T_{03}^2T_{04} + T_{11}^3T_{12}^2T_{13} + T_{21}^2T_{22} \rangle},$$

$$\tilde{Q} = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Remark 3.22.** Inserting the rays through  $(0, \dots, 0, \pm 1)$  as (partially) done in the preceding two examples is a “tropicalization step”: the two rays arise when we intersect the fan of the toric ambient variety with the tropicalization  $\text{trop}(X)$  which, in the example case, is the union of the one codimensional cones of the normal fan of the Newton polytope of the defining equation.

Combining the description in terms of data  $(A, P)$  with the language of bunched rings makes  $\mathbb{K}^*$ -surfaces an easily accessible class of examples. This allows among other things explicit classifications. For example, the Gorenstein del Pezzo  $\mathbb{K}^*$ -surfaces are classified by these methods in [24]; part of the results has been obtained by other methods in [16] and [41, 22].

**Theorem 3.23.** *The following table lists Cox ring  $\mathcal{R}(X)$  and the singularity type  $S(X)$  of the non-toric Gorenstein Fano  $\mathbb{K}^*$ -surfaces  $X$  of Picard number one.*

$\mathcal{R}(X)$	$\text{Cl}(X)$	$(w_1, \dots, w_r)$	$S(X)$
$\mathbb{K}[T_1, T_2, T_3, S_1] / \langle T_1^2 + T_2^2 + T_3^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$	$D_4 3A_1$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1T_2 + T_3^2 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$	$2A_3 A_1$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1T_2 + T_3^2 + T_4^3 \rangle$	$\mathbb{Z}$	$(1 \ 5 \ 3 \ 2)$	$A_4$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1T_2 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$A_5 A_1$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1T_2 + T_3^3 + T_4^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 \end{pmatrix}$	$A_5 A_2$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1T_2^2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$(4 \ 1 \ 2 \ 3)$	$D_5$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1^2T_2 + T_3^2 + T_4^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$	$D_6 A_1$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1T_2^2 + T_3^3 + T_4^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix}$	$E_6 A_2$

$\mathbb{K}[T_1, \dots, T_4] / \langle T_1 T_2^3 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$(3 \ 1 \ 2 \ 3)$	$E_6$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1 T_2^3 + T_3^4 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$E_7 A_1$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1 T_2^4 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$(2 \ 1 \ 2 \ 3)$	$E_7$
$\mathbb{K}[T_1, \dots, T_4] / \langle T_1 T_2^5 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$(1 \ 1 \ 2 \ 3)$	$E_8$
$\mathbb{K}[T_1, \dots, T_5] / \langle \begin{smallmatrix} T_1 T_2 + T_3^2 + T_4^2 \\ \lambda T_3^2 + T_4^2 + T_5^2 \end{smallmatrix} \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$2D_4$

**Theorem 3.24.** *The following table lists Cox ring  $\mathcal{R}(X)$  and the singularity type  $S(X)$  of the non-toric Gorenstein Fano  $\mathbb{K}^*$ -surfaces  $X$  of Picard number two.*

$\mathcal{R}(X)$	$\text{Cl}(X)$	$(w_1, \dots, w_r)$	$S(X)$
$\mathbb{K}[T_1, \dots, T_4, S_1] / \langle T_1 T_2 + T_3^2 + T_4^2 \rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$	$A_3 2A_1$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ -1 & 1 & 2 & -2 & 0 \end{pmatrix}$	$2A_2 A_1$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 3 & 1 & 3 & 2 \\ 1 & 1 & 0 & 2 & 1 \end{pmatrix}$	$A_2$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3 T_4 + T_5^3 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}$	$A_1 A_3$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3^2 T_4 + T_4^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 3 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}$	$A_3$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3^2 T_4 + T_5^3 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ -1 & 1 & 1 & -2 & 0 \end{pmatrix}$	$A_4 A_1$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2 + T_3^3 T_4 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 3 & 1 & 1 & 2 \\ -1 & -1 & 0 & -2 & -1 \end{pmatrix}$	$A_4$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^2 + T_3 T_4^2 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 2 & 1 & 2 & 1 & 2 \\ 0 & -1 & -2 & 0 & -1 \end{pmatrix}$	$D_4$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^2 + T_3 T_4^2 + T_5^3 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & -2 & 1 & 0 \end{pmatrix}$	$D_5 A_1$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^3 + T_3 T_4^2 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 & 2 & 0 & 2 \\ 1 & -1 & -2 & 0 & -1 \end{pmatrix}$	$D_5$
$\mathbb{K}[T_1, \dots, T_5] / \langle T_1 T_2^3 + T_3 T_4^3 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & -2 & 0 & -1 \end{pmatrix}$	$E_6$
$\mathbb{K}[T_1, \dots, T_6] / \langle \begin{smallmatrix} T_1 T_2 + T_3 T_4 + T_5^2 \\ \lambda T_3 T_4 + T_5^2 + T_6^2 \end{smallmatrix} \rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$	$2A_3$

**Theorem 3.25.** *The following table lists Cox ring  $\mathcal{R}(X)$  and the singularity type  $S(X)$  of the non-toric Gorenstein Fano  $\mathbb{K}^*$ -surfaces  $X$  of Picard number three.*

$\mathcal{R}(X)$	$\text{Cl}(X)$	$(w_1, \dots, w_r)$	$S(X)$
$\mathbb{K}[T_1, \dots, T_5, S_1] / \langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$A_1 A_2$
$\mathbb{K}[T_1, \dots, T_6] / \langle T_1 T_2 + T_3 T_4 + T_5 T_6 \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$	$3A_1$
$\mathbb{K}[T_1, \dots, T_6] / \langle T_1 T_2 + T_3 T_4 + T_5 T_6 \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$A_1$
$\mathbb{K}[T_1, \dots, T_6] / \langle T_1 T_2 + T_3 T_4 + T_5^2 T_6 \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$A_2$
$\mathbb{K}[T_1, \dots, T_6] / \langle T_1 T_2 + T_3 T_4^2 + T_5 T_6^2 \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$	$A_3$
$\mathbb{K}[T_1, \dots, T_6] / \langle T_1 T_2^2 + T_3 T_4^2 + T_5 T_6^2 \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$	$D_4$
$\mathbb{K}[T_1, \dots, T_7] / \langle \begin{smallmatrix} T_1 T_2 + T_3 T_4 + T_5 T_6 \\ \lambda T_3 T_4 + T_5 T_6 + T_7^2 \end{smallmatrix} \rangle$	$\mathbb{Z}^3$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$	$2A_2$

**Theorem 3.26.** *There is one non-toric Gorenstein Fano  $\mathbb{K}^*$ -surface of Picard number four; its Cox ring  $\mathcal{R}(X)$  and singularity type  $S(X)$  is given below.*

$\mathcal{R}(X)$	$\text{Cl}(X)$	$(w_1, \dots, w_r)$	$S(X)$
$\mathbb{K}[T_1, \dots, T_6, S_1] / \langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$	$\mathbb{Z}^4$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$A_1$

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